Hierarchical Random Operators & Matrices

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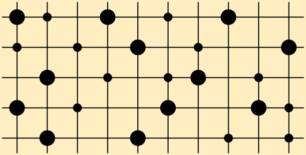
Motivation

Challenge: Understand the various facets of metal-insulator transitions in disordered systems, e.g.

Random Self-Adjoint Operator — Anderson '58

$$H(\omega) := \Delta + V(\omega)$$

e.g. on $\ell^2(\mathbb{Z}^d)$.



- Localization-delocalization transition of spectral-type/eigenfunctions with critical dimension $d = 2$.

  Goldsheid/Molchanov/Pastur '77, Fröhlich/Spencer '83, Simon/Wolf '84, . . . , Aizenman/Molchanov '92, ... Klein '94, Aizenman/W. '11

- Level statistics changes from Poisson to Random Matrix (e.g. GOE)

  Molchanov '81, Minami '96, . . .

Motivation

Challenge: Understand the various facets of metal-insulator transitions in disordered systems, e.g.

(Power Law) Random Band Matrices

$$\mathbb{E}\left[H_{jk}\right] = 0$$

$$\mathbb{E}\left[H_{jk}^2\right] = \begin{cases} 1 & \text{if } d(j,k) \leq W \\ 0 & \text{else} \end{cases} \quad \text{or} \quad = \begin{cases} 1 & \text{if } d(j,k) \leq 1 \\ \mathbf{d}(\mathbf{j},\mathbf{k})^{-\mathbf{2}\alpha} & \text{else} \end{cases}$$

e.g. real, symmetric $n \times n$ matrix (H_{jk})

- Localization-delocalization transition of eigenfunctions at $W = \sqrt{n}$ resp. $\alpha = 1$
- Level statistics changes from Poisson to Random Matrix (e.g. GOE)

Casati/Molinari/Izrailev '90, Fyodorov/Mirlin '91, . . .
Mirlin/Fyodorov/Dittes/Quezada/Seligman '96, , . . .
Schenker '09, . . . Erdös/Knowles/Yau/Yin '12, . . . , Shcherbyna '13, . . .

Dyson's advice

In view of the conjectured universality . . . . . . **go hierarchical!**

Aim: Retain crucial features while gaining simplicity
– **effective dimension** as tunable parameter !

Other areas:

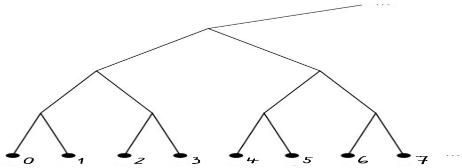
- Ising ferromagnet Dyson '69, Bleher/Sinai '73
- Φ^4 model Gawedzki/Kupiainen '83
- Directed polymers Derrida/Griffiths '89
- . . .

- *Here:* Anderson model & PRBM Bovier '90, Molchanov '96, . . . , Metz/Leuzzi/Parisi '13, . . . Fyodorov/Ossipov/Rodriguez '09, . . .

Hierarchical random operators & matrices

Hierarchy on set X *this talk:* $X = \{0,1,2,\dots\}$ or $\{0,1,2,\dots,2^n-1\}$

(Ultra-) Metric: $d(j,k) = \min\{r \mid j,k \text{ lie in the same partition of size } 2^r\}$

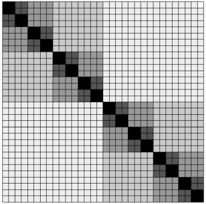


Nested sequence of partitions $\mathcal{P}_r$, $r \geq 0$

Hierarchical operator H on $\ell^2(X)$

$$H = \sum_{r\geq 0} \sum_{B\subset \mathcal{P}_r} H(B)$$

with self-adjoint $H(B)$ on $\ell^2(B)$.



Examples

1. **Hierarchical Laplacian:** $\mathbf{p} = (p_r) \in \ell^1$ *this talk:* $p_r = 2^{-cr}$, $c > 0$.

$$\Delta = \sum_{r \geq 1} p_r \sum_{B \in \mathcal{P}_r} E(B)$$

with averaging operator $(E(B)\psi)(j) = \frac{1}{2^r} \sum_{k \in B} \psi(k)$.

2. **Hierarchical Anderson model:** Bovier '90, Molchanov '96

$$H(\omega) = \Delta + V(\omega)$$

$$V(\omega) = \sum_{B \in \mathcal{P}_0} \omega_B \,, \qquad (\omega_j) \text{ iid random variables.}$$

3. **Ultrametric ensemble:** Fyodorov/Ossipov/Rodriguez '09

$$H_n = \sum_{r=0}^{n} 2^{-\frac{(1+c)}{2} r} \sum_{B \in \mathcal{P}_r} \Phi_B$$

with (Φ_B) independent GOE matrices of size 2^r.

General feature of hierarchical operators

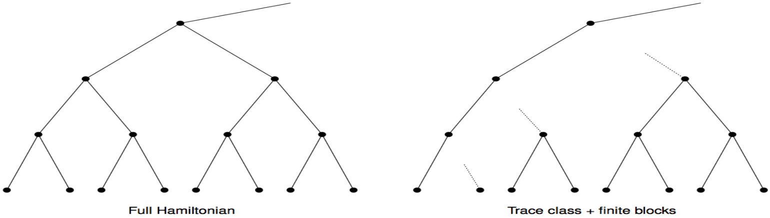
Close to direct sum: $H\delta_j = S_j\delta_j$ for all $j \in X$ with

$$S_j := \sum_{r \geq 0} H(B_r(j))$$

Theorem (Soosten/W. '16)

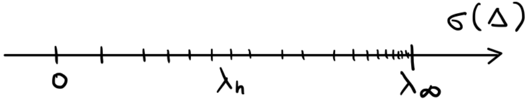
If S_j is trace class for some j, then $\sigma_{\mathrm{ac}}(H) = \emptyset$.

Proof idea: (Kato-Rozenblum thm, cf. Dombrowski '84-'85, Simon/Spencer '89)



Applications: no ac spectrum

1. **Hierarchical Laplacian**



Spectrum:

- infinitely degenerate ev's $0, \ldots, \lambda_n = \sum_{r=1}^{n} p_r, \ldots$
- eigenfunctions become delocalized as $\lambda_n \to \lambda_\infty$ ($|\mathrm{supp}\, \psi_n| = 2^n$)

2. **Hierarchical Anderson model**

- Using Simon-Wolff techniques (as in Kritchevski '07) one may even show that the spectrum of H is **almost surely pure point**.
- *Why is this surprising?*

The spectral dimension – a red herring and past conjectures

Spectral Dimension: i.e., local integrated DOS $\sim |\lambda - \lambda_\infty|^{\frac{d_s}{2}}$ as $\lambda \sim \lambda_\infty$

- **Hierarchical Laplacian** $p_r = 2^{-cr}$**:** $d_s = \frac{2}{c}$
- Random walk is recurrent resp. transient if $d_s \leq 2$ resp. $d_s > 2$!

Predictions & results on hierarchical Anderson model

- Conjectured critical dimension — Bovier '90
- Proof of only pure-point spectrum in special cases — Molchanov '96, Kritchevski '07
- Numerical evidence of extended states near λ_∞ in case $d_s > 2$ — Metz/Leuzzi/Parisi/Sacksteder '13-'14

Remaining questions:

Interesting effects in **finite volume**? **Level statistics**?

$$H_n = \sum_{r=1}^{n} p_r E_r 1_{B_n} + V\, 1_{B_n}$$

Results

Assumption: on energy interval $I \subset \mathbb{R}$ and probability density ϱ:

$$\exists \delta > 0 : \qquad \sup_{E \in I} \| T_{p_r} \cdots T_{p_1} \varrho(\cdot + E) \|_\infty = \mathcal{O}\left(2^{(c-\delta)r}\right)$$

where $p_r \sim 2^{-cr}$ and $T_p \varrho$ is the probability density of

$$\left(\frac{1}{2V} + \frac{1}{2V'}\right)^{-1} + p$$

with V and V' independently drawn from ϱ.

Proof of this assumption for *any* $I \subset \mathbb{R}$ in case:

- ϱ has a Cauchy component,
- ϱ is a Gaussian and $d_s < 4$,
- $d_s < 2$.

Numerical studies suggest that the assumptions is valid for all ϱ.!

Main result

Theorem (Soosten/W. '16)

Under the above assumption on $I \subset \mathbb{R}$ and probability density ϱ:

1. *the* **eigenfunction correlator**

$$Q_n(j,k;I) = \sum_{E \in I \cap \sigma(H_n)} |\psi_{E,n}(j)||\psi_{E,n}(k)|$$

is exponentially localized, i.e., for some $C, \mu \in (0,\infty)$

$$\sup_{n\in\mathbb{N}} \sup_{j\in\mathbb{N}_0} \sum_{k\in\mathbb{N}_0} 2^{\mu d(j,k)}\, \mathbb{E}[Q_n(j,k;I)] \le C|I|.$$

Implication for **inverse participation ratio**: $P(\psi) = \frac{\sum_x |\psi(x)|^4}{\left[\sum_x |\psi(x)|^2\right]^2}$.

There exists some $C < \infty$ such that for any $E \in I$, $W, \varepsilon > 0$ and$n \in \mathbb{N}_0$:

$$\mathbb{P}\left(\begin{array}{c} \text{There is } \psi \in \ell^2(B_n) \text{ with } H_n\psi = \lambda\psi \text{ and} \\ |\lambda - E| \le 2^{-n-1} W \text{ such that } P_2(\psi) \le \varepsilon^4 \end{array} \right) \le C\, W \varepsilon$$

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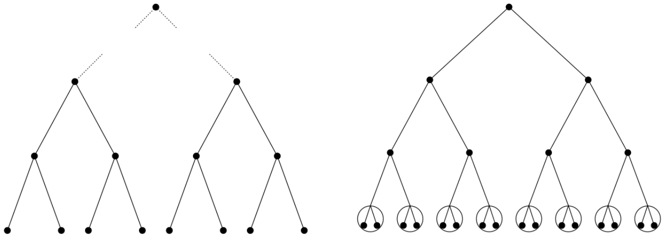
2. *the* **rescaled eigenvalue process**

$$\nu_n(f) = \sum_{\lambda \in \sigma(H_n)} f\left(2^n(\lambda - E)\right)$$

converges as $n \to \infty$ in distribution to a **Poisson point process** *with intensity given by the DOS.*



Philosophy I: Removing the root

Philosophy II: Collapsing the leaves

Krein-Feshbach-Schur (KFS) renormalization: (= Philosophy II !!!)

$$\mathcal{R} : \ell^1 \times L^1_+ \to \ell^1 \times L^1_+ \,, \qquad \mathcal{R}(\mathbf{p}, \varrho) = \big((p_{r+1})_{r \geq 1}, T_{p_1} \varrho \big) \ .$$

Renormalization on the Green function:

$$G_n(0, 2k, 0) = 2 \frac{V_1}{V_0 + V_1} \frac{V_{2k+1}}{V_{2k} + V_{2k+1}} \mathcal{R} G_{n-1}(0, k, 0)$$

Krein-Feshbach-Schur (KFS) renormalization:

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Renormalization on the Green function:

$$G_n(0, 2k, 0) = 2\frac{V_1}{V_0 + V_1}\,\frac{V_{2k+1}}{V_{2k} + V_{2k+1}}\,\mathcal{R}G_{n-1}(0, k, 0)$$

Proof idea:

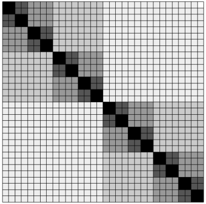
- Renormalization decreases the hopping strength $\mathcal{R}p_r = p_{r+1}$
- By assumption, the renormalization does not decrease the disorder strength
- After a finite number of steps an Aizenman-Molchanov type high disorder analysis applies.

Details: *Soosten/W., Renormalization Group Analysis of the Hierarchical Anderson Model, arXiv:1608.01602, to appear in AHP.*

Predictions & preliminary results on the ultrametric ensemble

Given (Φ_B), $B \in \mathcal{P}_r$, independent GOE matrices of size 2^r and parameter $c \in \mathbb{R}$:

$$H_n = \sum_{r=0}^{n} 2^{-\frac{(1+c)}{2} r} \sum_{B \in \mathcal{P}_r} \Phi_B$$



Predictions: Fyodorov/Ossipov/Rodriguez '09

- Localization-delocalization transition at $c = 0$
- Accompanying transition of level statistics from Poisson to GOE

Let A_0 be a symmetric $N \times N$ matrix (*here:* $N = 2^r$) and

$$\Phi_{jk}(t) = \frac{1}{\sqrt{N}} \begin{cases} \sqrt{2} B_{jj}(t) & j = k \\ B_{jk}(t) & j < k, \end{cases}$$

where $\{B_{jk}\}$ are independent standard BMs. The eigenvalues

$$A(t) = A_0 + \Phi(t)$$

evolve according to the SDE

$$d\lambda_k(t) = \sqrt{\frac{2}{N}} dB_k(t) + \frac{1}{N} \sum_{j \neq k} \frac{1}{\lambda_k(t) - \lambda_j(t)} dt.$$

Predicted **equilibration time** for **local eigenvalue statistics**: $t \gg N^{-1}$.

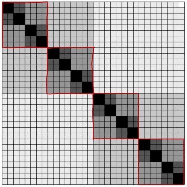
Landon/Yau '15, Landon/Sosoe/Yau '16

Here: $t = 2^{-(1+c)r}$ and hence equilibriation for $c < 0$!

The regime of Poisson statistics

Idea: Rescaled process

$$\nu_n(f) = \sum_{\lambda \in \sigma(H_n)} f\left(2^n(\lambda - E)\right)$$



is well approximated by 2^{n-m} **independent copies** of asymptotically negligible processes corresponding to

$$H_{n,m} = \sum_{r=0}^{m} 2^{-\frac{(1+c)}{2} r} \sum_{B \in \mathcal{P}_r} \Phi_B$$

Iteratively control $\operatorname{Tr} \frac{1}{H_n - z} - \operatorname{Tr} \frac{1}{H_{n,n-1} - z}$ at $\operatorname{Im} z \sim 2^{-n}$ using BM estimates

...

Theorem (Soosten/W. '17) *(Preliminary result)*

For $c > 1$ the **rescaled eigenvalue process** ν_n *converges as $n \to \infty$ in distribution to a* **Poisson point process** *with intensity given by the DOS.*