Geometry and temperature chaos in some spherical spin glasses

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Based on an ongoing work with Gerard Ben Arous and Ofer Zeitouni

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$$H_{N,p}(\mathbf{x}) = \sqrt{N} \sum_{i_1,\ldots,i_p=1}^{N} J_{i_1,\ldots,i_p} x_{i_1} x_{i_2} \cdots x_{i_p}, \quad \mathbf{x} \in \mathbb{S}^N.$$

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- **Covariance:** $\mathbb{E}\{H_{N,\nu}(\mathbf{x}) H_{N,\nu}(\mathbf{x}')\} = N\nu(\langle \mathbf{x}, \mathbf{x}' \rangle)$.

The Gibbs measure – general results

Gibbs meas.: $G_{N,\beta}(A) = \frac{1}{Z_{N,\beta}} \int_A e^{\beta H_N(\mathbf{x})} d\mathbf{x}, \qquad A \subset \mathbb{S}^N, \ \beta \geq 0.$

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Ultrametricity: (Panchenko '13)

$$\forall \epsilon > 0: \; G_{N,\beta}^{\otimes 3}\{d(\mathbf{x}_1, \mathbf{x}_3) \leq d(\mathbf{x}_1, \mathbf{x}_2) \vee d(\mathbf{x}_2, \mathbf{x}_3) + \epsilon\} \xrightarrow[N\to\infty]{P} 1.$$

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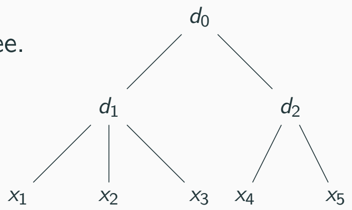
Remark: any finite ultrametric space $\{x_1, ..., x_k\}$ can be represented by a tree.



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Cluster decomposition: (Talagrand ‘10, Jagannath ‘14)

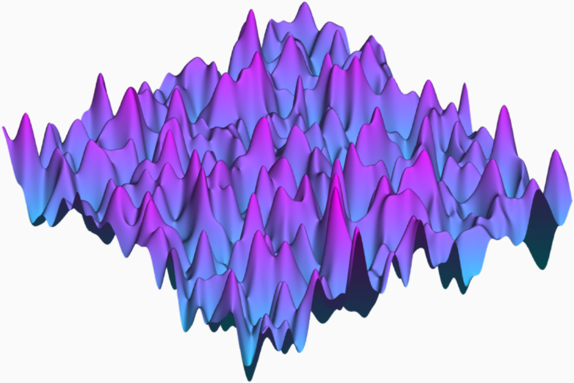
There exist disjoint random $A_i \subset \mathbb{S}^N$ such that:

1. $G_{N,\beta}\{\cup A_i\} \to 1$,
2. $\{G_{N,\beta}\{A_i\}\}_i$ converges in distribution to some $W_i > 0$,
3. $\exists t_{ij}$ random, s.t conditional on $\mathbf{x}_1 \in A_i, \mathbf{x}_2 \in A_j$,

$$G_{N,\beta}^{\otimes 2}\{|d(\mathbf{x}_1, \mathbf{x}_2) - t_{ij}| > \epsilon\} \to 0.$$

Critical points and the Gibbs measure, pure models

Critical points



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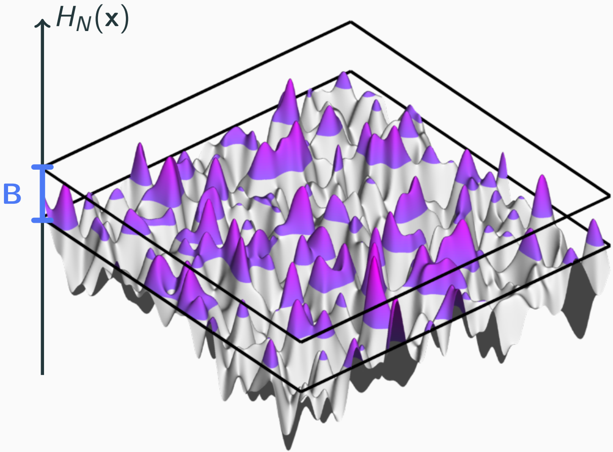

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For $B \subset \mathbb{R}$, $\quad \mathrm{Crit}(B) :=$ # critical values of $H_N(\mathbf{x})$ in B.

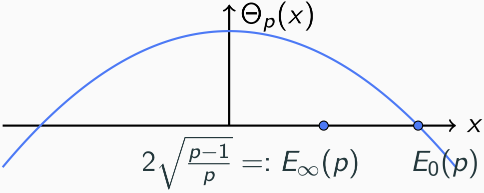
Critical points

Theorem (Auffinger–Ben Arous–Černý '13)

$$\lim_{N\to\infty} \frac{1}{N} \log\left(\mathbb{E}\left\{\mathrm{Crit}(NB)\right\}\right) = \sup_{x\in B} \Theta_p\left(x\right).$$

Theorem (S. '15)

For $B \subset (E_\infty, E_0)$, $$\frac{\mathrm{Crit}\left(NB\right)}{\mathbb{E}\left\{\mathrm{Crit}\left(NB\right)\right\}} \longrightarrow 1, \quad \text{in prob.}$$



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Lemma (S. '15)

$B_N\subset(E_\infty,E_0)$, $|B_N|\to 0 \implies \mathbb{P}\Big\{|\langle \mathbf{x}_1,\mathbf{x}_2\rangle|>\epsilon\Big\}\to 0$,
if $\mathbf{x}_1\neq\mathbf{x}_2$ are random (unif.) crt. pts. such that $H_N(\mathbf{x}_i)\in NB_N$.

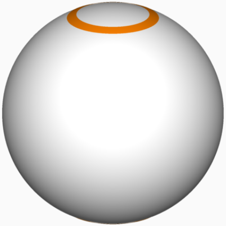
Theorem (S.–Zeitouni '16)

The critical values near $\mathbb{E}\max_{\mathbf{x}} H_N(\mathbf{x})$ converge to PPP$(e^{c_p x}dx)$.

The Gibbs measure

- Fix some sequence $N^{-\frac{1}{2}} \ll t_N \ll 1$.
  $q_* = q_*(\beta)$ explicit.

$$\mathrm{Band}\,(\mathbf{x}_0) := \left\{ \mathbf{x} \in \mathbb{S}^N : q_* - t_N \leq \langle \mathbf{x}, \mathbf{x}_0 \rangle \leq q_* + t_N \right\}.$$



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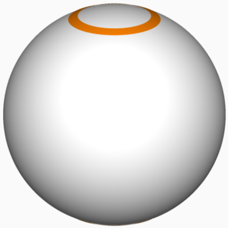
- Enumerate the crt. pts. $\mathbf{x}_0^i$, $i \ge 1$,
  in decreasing order:

$$H_N(\mathbf{x}_0^i) \ge H_N(\mathbf{x}_0^{i+1}).$$



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1. Asymptotic support: $\forall \epsilon > 0$, for large k, N,

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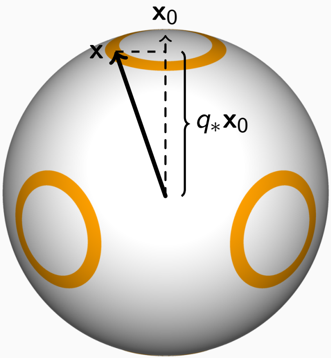
2. Suppose $\mathbf{x}_1, \mathbf{x}_2$ drawn independently from the Gibbs measure. Then, with probability going to 1, for small $\epsilon > 0$

$$|\langle \mathbf{x}_1, \mathbf{x}_2 \rangle| < \epsilon \iff \mathbf{x}_1, \mathbf{x}_2 \in \text{different bands},$$
$$|\langle \mathbf{x}_1, \mathbf{x}_2 \rangle - q_*^2| < \epsilon \iff \mathbf{x}_1, \mathbf{x}_2 \in \text{same band}.$$

Overlap distribution

- Given a sample $\mathbf{x} \in \mathrm{Band}(\mathbf{x}_0)$,

$$\mathbf{x} \approx q_* \mathbf{x}_0 + \mathbf{x}^\perp,$$
$$\langle \mathbf{x}_0, \mathbf{x}^\perp \rangle = 0.$$



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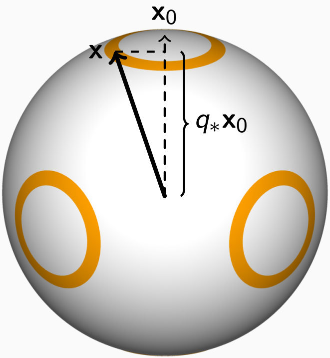

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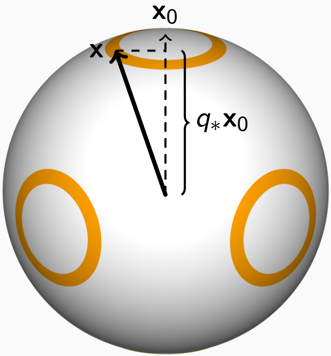
- Recall: deep crt. pts. are orthogonal w.h.p. Thus,

$$\langle \mathbf{x}_0, \mathbf{y}_0 \rangle \approx 0 \text{ or } 1.$$



Mixed models – work in progress (\w Ben Arous, Zeitouni)

- Reminder: $H_{N,\nu}(\mathbf{x}) = \sum_p \gamma_p H_{N,p}(\mathbf{x})$, $\nu(x) = \sum \gamma_p x^p$.

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- All the results about critical points hold as in the pure case.
- For large β, the Gibbs measure $G_{N,\beta}$ concentrates on bands around critical points.
- However, the relevant critical points depend on β!
- Compared to the pure case, more complicated structure on bands. For two samples from the same band $\langle \mathbf{x}_1, \mathbf{x}_2 \rangle > q_*^2$. But still, for points from different bands $\langle \mathbf{x}_1, \mathbf{x}_2 \rangle \approx 0$.

Temperature chaos

- Let $\beta_1 \neq \beta_2$ be two inverse-temperatures.
- With the disorder $H_N(\mathbf{x})$ fixed, let $\mathbf{x}_1$ and $\mathbf{x}_2$ be independent samples from G_{N,β_1} and G_{N,β_2}. (recall: $dG_{N,\beta} \propto e^{\beta H_N(\mathbf{x})} d\mathbf{x}$)

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- Small change in temperature results in a drastic change of the Gibbs measure.

Temperature chaos

Theorem (Chen–Panchenko '16)

For even generic models (under some conditions) temperature chaos **occurs.**

Even: p odd $\implies \gamma_p = 0$, generic: $\sum p^{-1}\mathbf{1}\{\gamma_p \neq 0\} = \infty$.

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Let $p \geq 4$ even, $p' \neq p$, $a \in (0, \frac{1}{4})$. For

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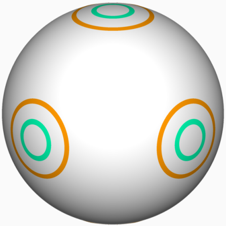
Sketch of proof. For large K, both G_{N,β_1} and G_{N,β_2} are essentially supported on the set of bands around $\mathbf{x}_0^1, ..., \mathbf{x}_0^K$ with $q_*(\beta_1) \neq q_*(\beta_2)$.

As before, if $\mathbf{x} \in \mathrm{Band}_{\beta_1}(\mathbf{x}_0^i)$, $\mathbf{x}' \in \mathrm{Band}_{\beta_2}(\mathbf{x}_0^j)$, then

$$\langle \mathbf{x}, \mathbf{x}' \rangle \approx q_*(\beta_1) q_*(\beta_2) \langle \mathbf{x}_0^i, \mathbf{x}_0^j \rangle.$$



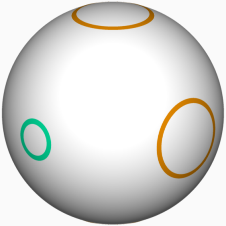
$$\Longrightarrow G_{N,\beta_1} \times G_{N,\beta_2} \left\{ \langle \mathbf{x}, \mathbf{x}' \rangle \approx q_*(\beta_1) q_*(\beta_2) \right\}$$
$$= \sum_i G_{N,\beta_1}(\mathrm{Band}_{\beta_1}(\mathbf{x}_0^i)) G_{N,\beta_2}(\mathrm{Band}_{\beta_2}(\mathbf{x}_0^i)).$$

On the other hand, for the mixed models (close to pure), the critical points G_{N,β_1} and G_{N,β_2} are supported correspond to different heights $NE_*^{\beta_1}$ and $NE_*^{\beta_2}$ of the Hamiltonian and they are orthogonal.

Therefore, typically for two samples $\mathbf{x} \in \mathrm{Band}_{\beta_1}(\mathbf{x}_0^i)$, $\mathbf{x}' \in \mathrm{Band}_{\beta_2}(\mathbf{x}_0^j)$,

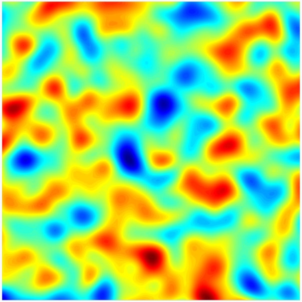
$$\langle \mathbf{x}, \mathbf{x}' \rangle \approx 0,$$

and temperature chaos occurs.



Conditional models on bands

- The Gibbs measure $dG_{N,\beta} \propto e^{\beta H_N(\mathbf{x})} d\mathbf{x}$ mainly charges high values of $H_N(\mathbf{x})$.



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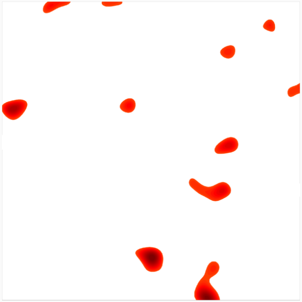

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- The complement is covered by nbhds
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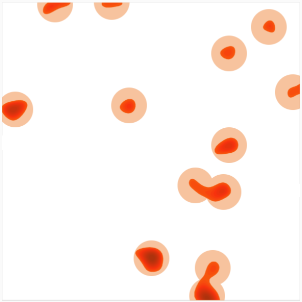
of critical pts $\mathbf{x}_0$ with $H_N(\mathbf{x}_0) \geq NE_t$.



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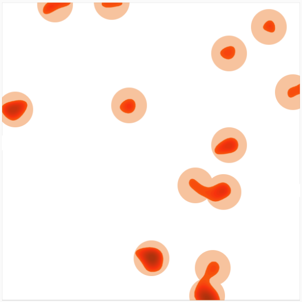

- The latter set is covered by the union, going over all heights $E > E_t$ and 'overlaps' $q \in (q_t, 1)$, of the bands:

$$\mathrm{Band}(\mathbf{x}_0, q) := \{\mathbf{x} : \langle \mathbf{x}, \mathbf{x}_0 \rangle \approx q\}, \quad \mathbf{x}_0 \text{ crit.}, \quad H_N(\mathbf{x}_0) \approx NE.$$

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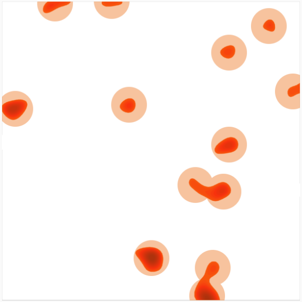

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- To prove that $G_{N,\beta}$ concentrates on bands, we show that the weight of those bands is maximized at log. scale for specific E_*, q_*.

Conditional models on bands

For given height E and 'overlap' q:

- Number of bands $\approx \exp\{N\Theta_\nu(E)\}$.
- Each band has volume $\approx (1-q^2)^{\frac{N}{2}}$.
- For each critical $\mathbf{x}_0$, associate the weight

$$\int e^{\beta H_N|_q(\mathbf{x})} d\mathbf{x},$$

where

$$H_N|_q : \mathbb{S}^{N-1} \to \mathbb{R}, \quad H_N|_q(\mathbf{x}) = H_N \circ f(\mathbf{x})$$

is the restriction of H_N to the sub-sphere $\{\mathbf{x} : \langle \mathbf{x}, \mathbf{x}_0 \rangle = q\}$, up to a change of coordinates.

Conditional models on bands

For given height E and 'overlap' q:

- Number of bands $\approx \exp\{N\Theta_\nu(E)\}$.
- Each band has volume $\approx (1-q^2)^{\frac{N}{2}}$.
- For each critical $\mathbf{x}_0$, associate the weight

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Simplification: assume those weights are i.i.d., with law obtained from conditioning on $\nabla H_N(\mathbf{x}_0) = 0\,,\ H_N(\mathbf{x}_0) = NE$.

Conditional models on bands – pure case

- For the pure p-spin model,

$$\begin{aligned} H_{N,p}|_q(\mathbf{x}) \overset{d}{=} &\ \alpha_{p,0}(q) H_{N,p}(\mathbf{x}_0) \\ &+ \alpha_{p,1}(q)\langle \nabla H_{N,p}(\mathbf{x}_0), \mathbf{x}\rangle \\ &+ \sum_{k=2}^{p} \alpha_{p,k}(q) H_{N,k}^{(p)}(\mathbf{x}), \end{aligned}$$

$H_{N,k}^{(p)}$ are copies of pure k-spin models,
$\alpha_{p,k}(q) \in \mathbb{R}$ explicit,
all summands are independent.

- Upon conditioning on $H_{N,p}(\mathbf{x}_0) = NE$, $\nabla H_{N,p}(\mathbf{x}_0) = 0$,

$$H_{N,p}|_q(\mathbf{x}) \overset{d}{=} \alpha_{p,0}(q) NE + \sum_{k=2}^{p} \alpha_{p,k}(q) H_{N,k}^{(p)}(\mathbf{x}).$$

Conditional models on bands – pure case

$$H_{N,p}|_q(\mathbf{x}) \stackrel{d}{=} \alpha_{p,0}(q)NE + \sum_{k=2}^{p} \alpha_{p,k}(q)H_{N,k}^{(p)}(\mathbf{x})$$

- The weight corresponding to a single band is

$$\int e^{\beta H_{N,p}|_q(\mathbf{x})} d\mathbf{x} = e^{N\alpha_{p,0}(q)\beta E} \int e^{\beta \sum_{k=2}^{p} \alpha_{p,k}(q) H_{N,k}^{(p)}(\mathbf{x})} d\mathbf{x}.$$

- Deterministic, only term that depends on E.
- Distributed like $e^{N(C+W)}$, for $w \neq 0$, $\mathbb{P}\{W \approx w\} = e^{-N^2 J(w)}$, C deterministic – essentially can be replaced by mean.
- For large β always best to choose $E = E_0$, lose number of points and gain in the first term above. $\implies E_* = E_0$.
- Optimize over q to find q_*.

Conditional models on bands – mixed case

- In the mixed case $\left(H_{N,\nu}(\mathbf{x}) = \sum_{p\geq 2} \gamma_p H_{N,p}(\mathbf{x})\right)$,

$$H_{N,\nu}|_q(\mathbf{x}) \overset{d}{=} \sum_{p\geq 2} \gamma_p \alpha_{p,0}(q) H_{N,p}(\mathbf{x}_0) + \left\langle \sum_{p\geq 2} \gamma_p \alpha_{p,1}(q) \nabla H_{N,p}(\mathbf{x}_0), \mathbf{x} \right\rangle + \sum_{p\geq 2} \sum_{k=2}^{p} \gamma_p \alpha_{p,k}(q) H_{N,k}^{(p)}(\mathbf{x}).$$

- Upon conditioning on

$$H_{N,\nu}(\mathbf{x}_0) = \sum_{p\geq 2} \gamma_p H_{N,p}(\mathbf{x}_0) = NE,$$

$$\nabla H_{N,\nu}(\mathbf{x}_0) = \sum_{p\geq 2} \gamma_p \nabla H_{N,p}(\mathbf{x}_0) = 0,$$

non of the terms becomes deterministic.

Conditional models on bands – mixed case

• The weight of one band $\approx\ e^{N(C+W)}$,
$C > 0$ deterministic, for $w \neq 0$, $\mathbb{P}\{W \approx w\} = e^{-NJ(w)}$.

• However, there are exponentially many points if $E < E_0$, leading to a large deviation type problem.

• The optimal energy E_* turns out to be strictly smaller than E_0 and β dependent.

Disorder chaos – pure models

Disorder chaos

- Let $H'_N(\mathbf{x})$ be an i.i.d. copy of $H_N(\mathbf{x})$ and for $t \in (0,1)$ set

$$H_{N,t}(\mathbf{x}) := (1-t)H_N(\mathbf{x}) + \sqrt{2t - t^2}H'_N(\mathbf{x}).$$

- Let $\mathbf{x}_1$ and $\mathbf{x}_2$ be independent samples from $G_{N,\beta}$ and $G_{N,t,\beta}$.

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- We say that **disorder chaos** occurs if

$$\langle \mathbf{x}_1, \mathbf{x}_2 \rangle \xrightarrow[N\to\infty]{P} 0.$$

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Theorem (Chen–Hsieh–Hwang–Sheu '15)

Disorder chaos occurs for all spherical models and $t \in (0,1)$.

Disorder chaos

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Theorem (S.–Zeitouni '16)

There exists a random permutation $\sigma : \mathbb{N} \to \mathbb{N}$ such that for fixed K, for any $i < K$:

1. Location hardly changes: $\|\mathbf{x}_0^i - \mathbf{x}_0^{\sigma(i)}(t_N)\| = o(1)$.

2. Change in value: $H_{N,t_N}(\mathbf{x}_0^{\sigma(i)}(t_N)) = H_N(\mathbf{x}_0^i) + \Delta_i$,

$$\Delta_i := -Nt_N C + \sqrt{Nt_N}\frac{H_N'(\mathbf{x}_0^i)}{\sqrt{N}} + o(1)\,.$$

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- Bands approximately remain at same position, weights change.

Disorder chaos

$$\Delta_i = -Nt_N C + \sqrt{Nt_N}\frac{H'_N(\mathbf{x}_0^i)}{\sqrt{N}} + o(1)$$

- For $t_N = c/N$, for any $i \leq K$, w.h.p. $\Delta_i = O(1)$.

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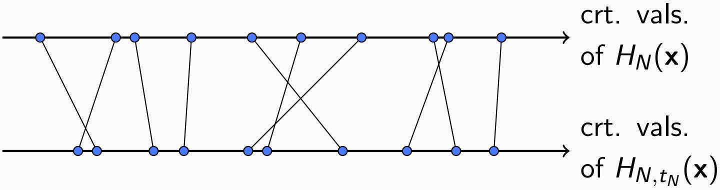

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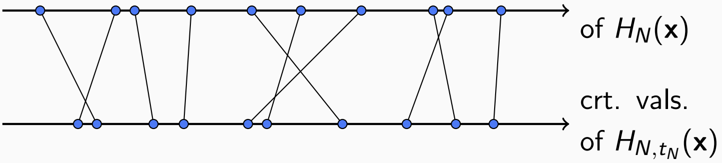


- No disorder chaos.

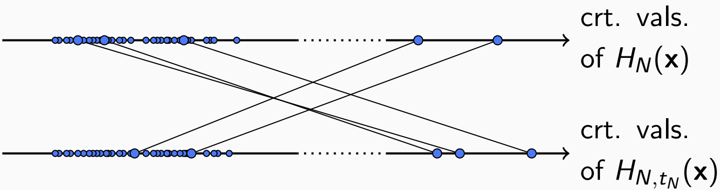
Disorder chaos

$$\Delta_i = -Nt_N C + \sqrt{Nt_N}\frac{H'_N(\mathbf{x}_0^i)}{\sqrt{N}} + o(1)$$

- If $t_N = c_N/N$ with $c_N \to \infty$, for $i \leq K$,

$$Nt_N C \gg \left|\sqrt{Nt_N}\frac{H'_N(\mathbf{x}_0^i)}{\sqrt{N}}\right|.$$

- But $H_N(\mathbf{x}) \stackrel{d}{=} H_{N,t_N}(\mathbf{x})$, not all points are washed away by shift.



- Disorder chaos occurs.

Thank You!