Weizmann 24.1.2017

KPZ Growing Interfaces: How Flat is Flat?

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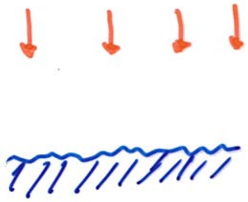
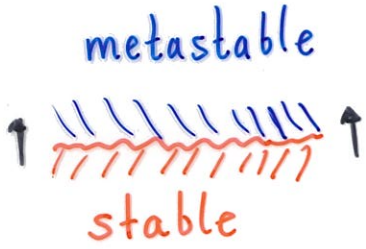
joint work S. Chhita, P.L. Ferrari (Bonn)

KPZ growth



deposition



- independet nucleations
- diffusive relaxation
- sidewards growth

HERE 1+1 dimensions

height function h

$$\partial_t h = \frac{1}{2}\lambda(\partial_x h)^2 + \frac{1}{2}\partial_x^2 h + W$$

space-time white noise

$\Rightarrow$ flat initial data $h(x,0) = 0 \Leftarrow$

Conjecture:

$$h(0,t) \cong v_0 t + (\Gamma t)^{1/3}\, \xi_{GOE}$$

model

universal

$$\mathbb{P}(\xi_{GOE} \leq s) = \det(1 - P_s K^0 P_s)$$

$$K^0(x,y) = \mathrm{Ai}(x+y)$$

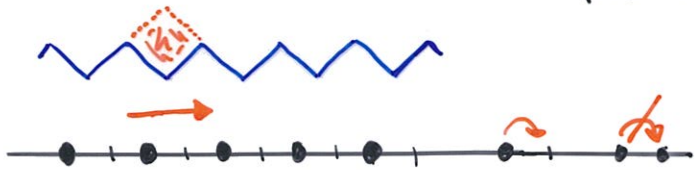
P_s projects on $[s,\infty)$

proved in discrete setting

- single step growth

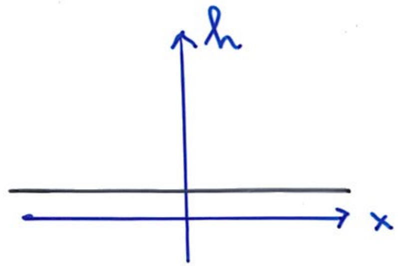
  TASEP

fill minima with rate 1

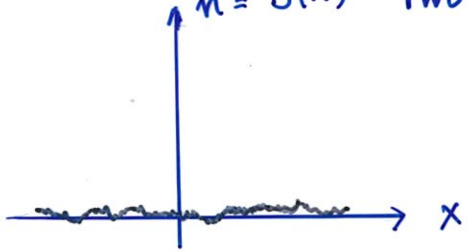


Borodin, Ferrari, Prähofer, Sasamoto 2006

How flat is flat?



$h = B(x)$ two-sided BM



Theorem (Borodin, Corwin, Ferrari, Vetö 2014)

$$h(0,x) = B(x)$$

$$h(0,t) \cong v_0 t + (\Gamma t)^{1/3} \underbrace{\xi_{BR}}_{\text{Baik-Rains}}$$

What about

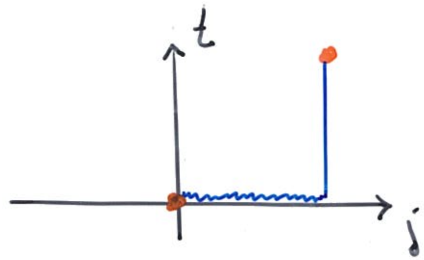
$$\langle h(x,0) \rangle = 0$$

$$\langle h(x,0)^2 \rangle \cong |x|^{2\kappa} \qquad \kappa < 1 \quad ?$$

$\Rightarrow$ $h'(x)$ stationary

single step

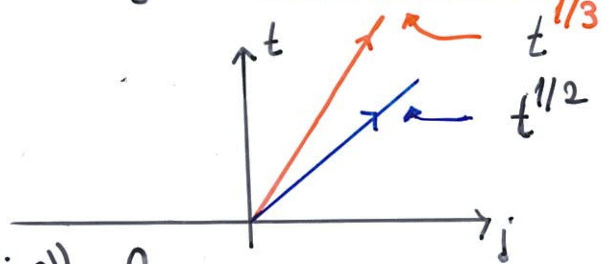
$\eta_j = \begin{cases} 0 \\ 1 \end{cases}, \; j \in \mathbb{Z}$, $\eta_j(t)$ $\Rightarrow$ height $h(j,t)$



- stationary $\{\eta_j\}$ Bernoulli $\mathbb{E}(\eta_j) = \rho$ Ferrari, Hs. 2006

- $\rho = \frac{1}{2}$ $h(0,t) \simeq v_0 t + (\Gamma t)^{1/3} \xi_{BR}$

- general ρ : current $j(\rho) = \rho(1-\rho)$, $j'(\rho) = 1 - 2\rho$ characteristic



HERE

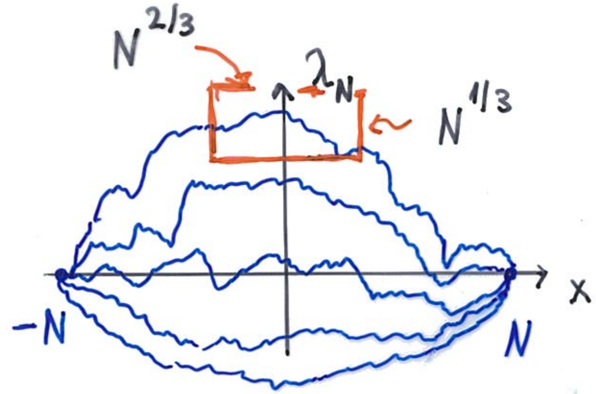
$j \to \eta_j$ stationary process $\mathbb{E}(\eta_j) = \frac{1}{2}$, $\mathbb{E}(h(j,0)) = 0$

- CLT $\lim_{\ell \to \infty} \frac{1}{\sqrt{\ell}} h(\lfloor \gamma \ell x \rfloor, 0) = \sqrt{2}\, \sigma B(x)$ two-sided Brownian motion

weak convergence of path measure on $C(\mathbb{R})$

a variational result (Johansson 2003)

Airy$_2$ process $\mathcal{A}i(x)$



$$\lim_{N\to\infty} N^{-1/3}\left(\lambda_N(N^{2/3}x) - 2N\right) = \mathcal{A}i(x) - x^2$$

N Brownian bridges

conditioned on nonintersection

$\Rightarrow \mathcal{A}i(x)$ continuous path
stationary
locally Brownian
fixed x : $\mathcal{A}i(x)$ is GUE Tracy-Widom
long-ranged $\mathbb{E}\left(\mathcal{A}i(x)\,\mathcal{A}i(0)\right)_c \cong \frac{1}{|x|^2}$

Theorem

$$\sup_x \left\{ \mathcal{A}i(x) - x^2 \right\} \text{ is GOE Tracy-Widom}$$

stationary initial data

TASEP

$$F^{(\sigma)}(s) = \mathbb{P}\left(\sup_u \left\{ \sqrt{2}\,\sigma B(u) + \mathcal{A}i(u) - u^2 \right\} \leq s \right)$$

// universal

Theorem

$$\lim_{t\to\infty} \mathbb{P}\left(h(0,t) \geq \tfrac{1}{2} t - s(t/2)^{1/3} \right) = F^{(\sigma)}(s)$$

⟦ Flat is $\sigma = 0$ ⟧

subdiffusive

deterministic initial data, KPZ

Quastel, Remenik 2016

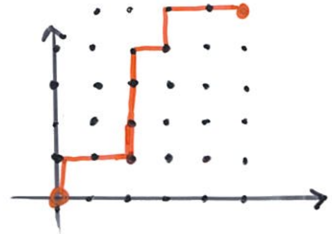
special cases

$\sigma = 0$ GOE Tracy-Widom

$\sigma = 1$ Baik-Rains this theorem + Ferrari, HS 2006

$\sigma = \infty$ $\mathbb{P}\left(\sup_u \{ B(u) - u^2 \} \leq s \right)$ Groeneboom 2010

Proof relies on TASEP $\Leftrightarrow$ last passage percolation



i.i.d. EXP up/right

optimal path

tightness

slow decorrelation

Cator, Pimentel 2015

Corwin, Ferrari, Peché 2012

Borodin, Peché 2008

Corwin, Liu, Wang 2016

HERE: formal argument based on KPZ

sketch

$$\partial_t h = \tfrac{1}{2}(\partial_x h)^2 + \tfrac{1}{2}\partial_x^2 h + W \qquad \| \qquad h(x,0) = \sigma B(x)$$

stationary: $\sigma = 1$

Funaki, Quastel 2015

Cole-Hopf $\quad Z = e^{h}$

$$\partial_t Z = \tfrac{1}{2}\partial_x^2 Z + W Z$$

directed polymer $b(t)$

$$Z(x,t) = \mathbb{E}_{b(t)=x}\left(e^{\int_0^t W(b(s),s)\,ds}\, e^{\sigma B(b(0))} \right)$$

finite temperature
last passage percolation

random

$$Z(0,t) = \int dx\ \underbrace{\mathbb{E}_{b(t)=0,\, b(0)=xt^{2/3}}\left(e^{\int_0^t W(b(s),s)\,ds} \right)}_{e^{t^{1/3}(Ai(x)-x^2)+v_0 t}} e^{\sigma B(xt^{2/3})}$$

point-to-point
fixed x theorem

$\rightsquigarrow$

$$Z(0,t) \cong e^{v_0 t}\int dx\ e^{t^{1/3}(Ai(x) - x^2 + \sigma B(x))}$$

$$\lim_{t\to\infty} t^{-1/3}\left(\log Z(0,t) - v_0 t\right) = \sup_u \left\{ Ai(u) - u^2 + \sigma B(u) \right\}$$

→ properties of $F^{(\sigma)}$? Monte-Carlo simulations

→ universality conical (KPZ) point of 6-vertex model

MC of TASEP

$\eta_j = \begin{cases} 0 \\ 1 \end{cases}$ Markov chain $0 \Leftrightarrow 1$ probability $1-\alpha$

α	σ
0	0
$\frac{1}{2}$	1
$\to 1$	$\to \infty$

NO adjustable parameters

- large σ approximation

$$F^{(\sigma)}_{app}(s) = \mathbb{P}\big(\underbrace{Ai(0)}_{GUE} + \sup_u \{\sqrt{2}\,\sigma\, B(u) - u^2\} \leq s\big)$$

from Groeneboom 2010

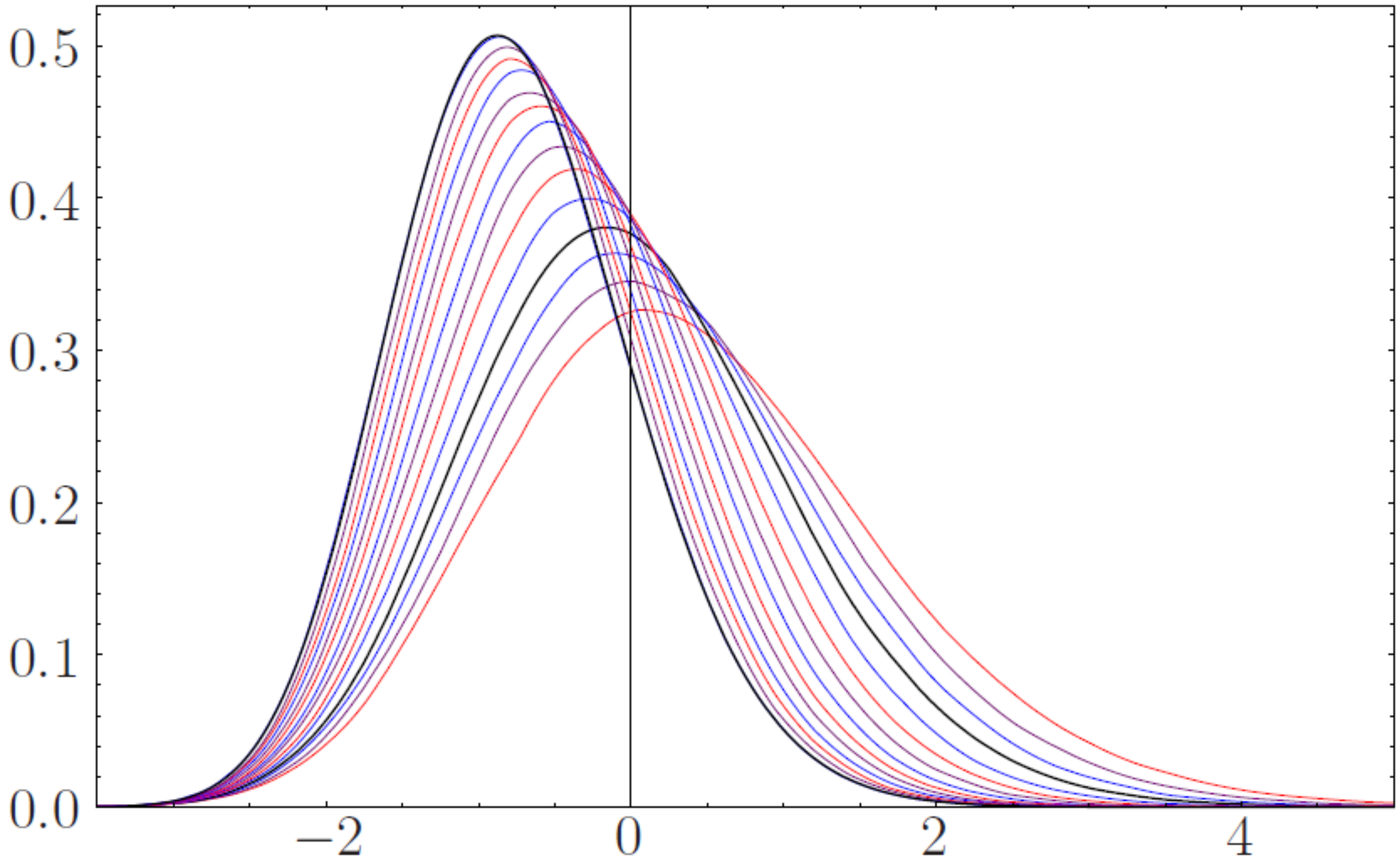


Figure 6: Probability densities of $F^{(\sigma)}(s)$ with $\sigma = \sqrt{\alpha/(1-\alpha)}$ from TASEP simulation until time $t_{\mathrm{max}} = 10^3$ and 10^6 runs. The different plots corresponds to the values $\alpha = 0, 0.05, 0.1, 0.15, 0.20, 0.25, 0.3, 0.35, 0.4, 0.45, 0.5, 0.54, 0.58, 0.62$. The left-most black line is the exact rescaled GOE distribution ($\sigma = 0$), which overlaps with $\alpha = 0$ from the simulations. The black line in the middle is the stationary case ($\sigma = 1$).

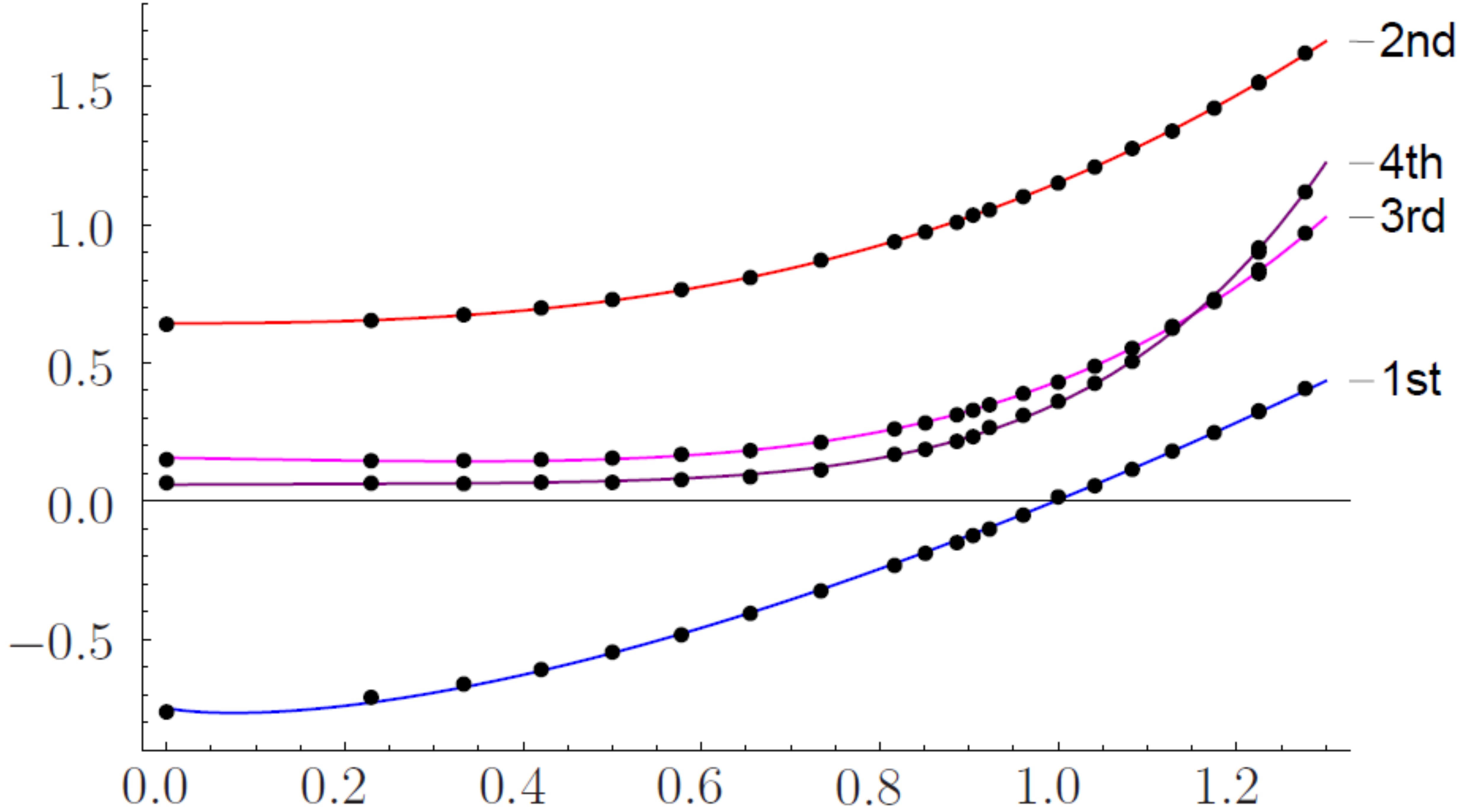


Figure 5: Empirical values for the first four cumulants of $F^{(\sigma)}$ as a function of σ. The empirical extrapolations are: $\kappa_1(\sigma) = -0.74795 - 0.96976\sigma + 1.7217\sigma^{4/3}$, $\kappa_2(\sigma) = 0.64268 + 0.0068163\sigma + 0.50202\sigma^{8/3}$, $\kappa_3(\sigma) = 0.15631 - 0.049956\sigma + 0.32731\sigma^4$, and $\kappa_4(\sigma) = 0.059476 + 0.012039\sigma + 0.28315\sigma^{16/3}$. The fitting made for the values shown in the figure, gives quite good results also for large values of σ

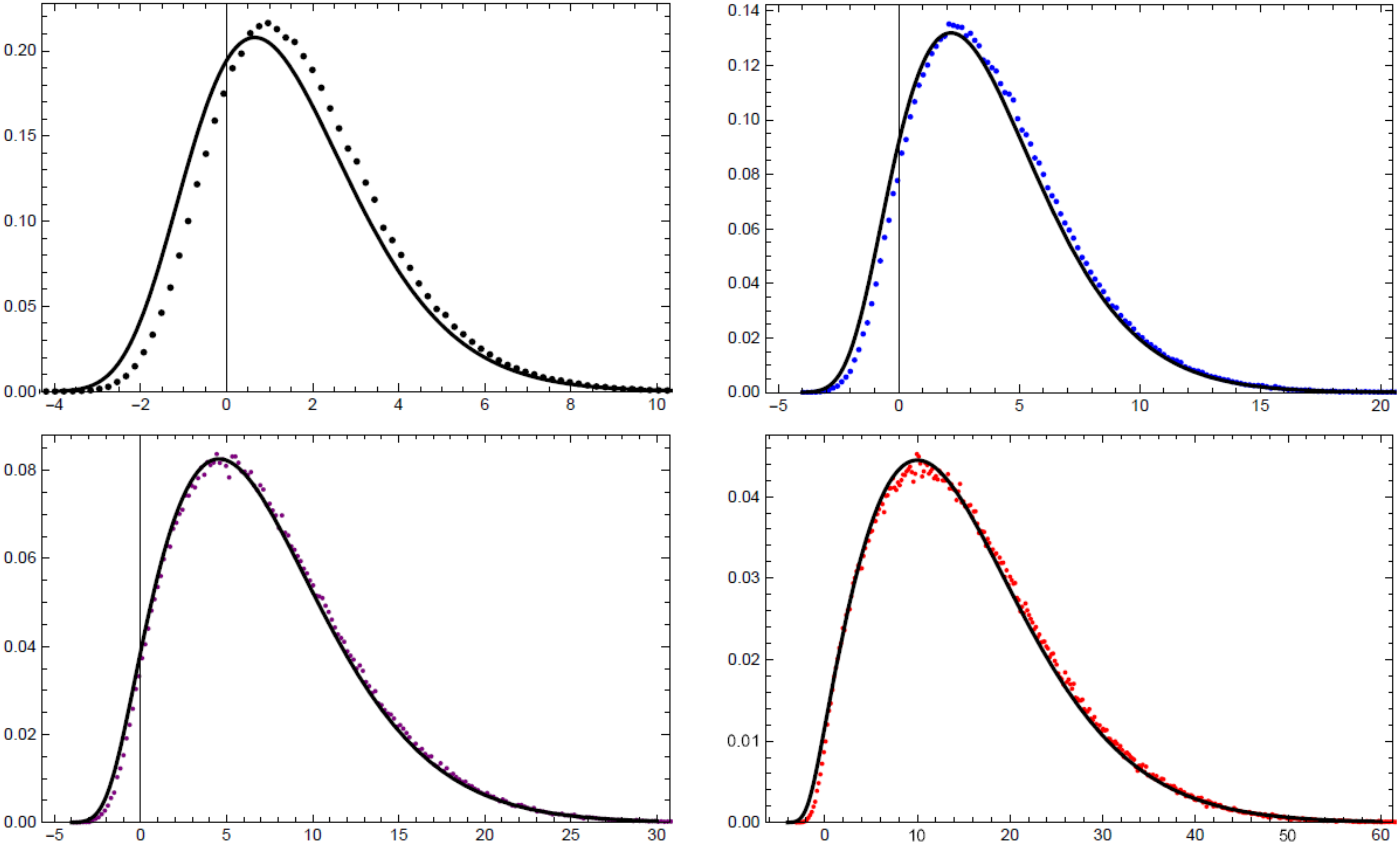


Figure 3: Comparison between the probability densities of $F^{(\sigma)}(s)$ and $F^{(\sigma)}_{\mathrm{appr}}(s)$: top-left is $\sigma = 2.0$, top-right is $\sigma = 3.0$, bottom-left is $\sigma = 4.36$ and bottom-right is $\sigma = 7.0$. The simulation for $\sigma = 2.0$ uses maximal time $t_{\max} = 2000$, while the other $t_{\max} = 3000$. The number of runs are 5×10^5 for each case.

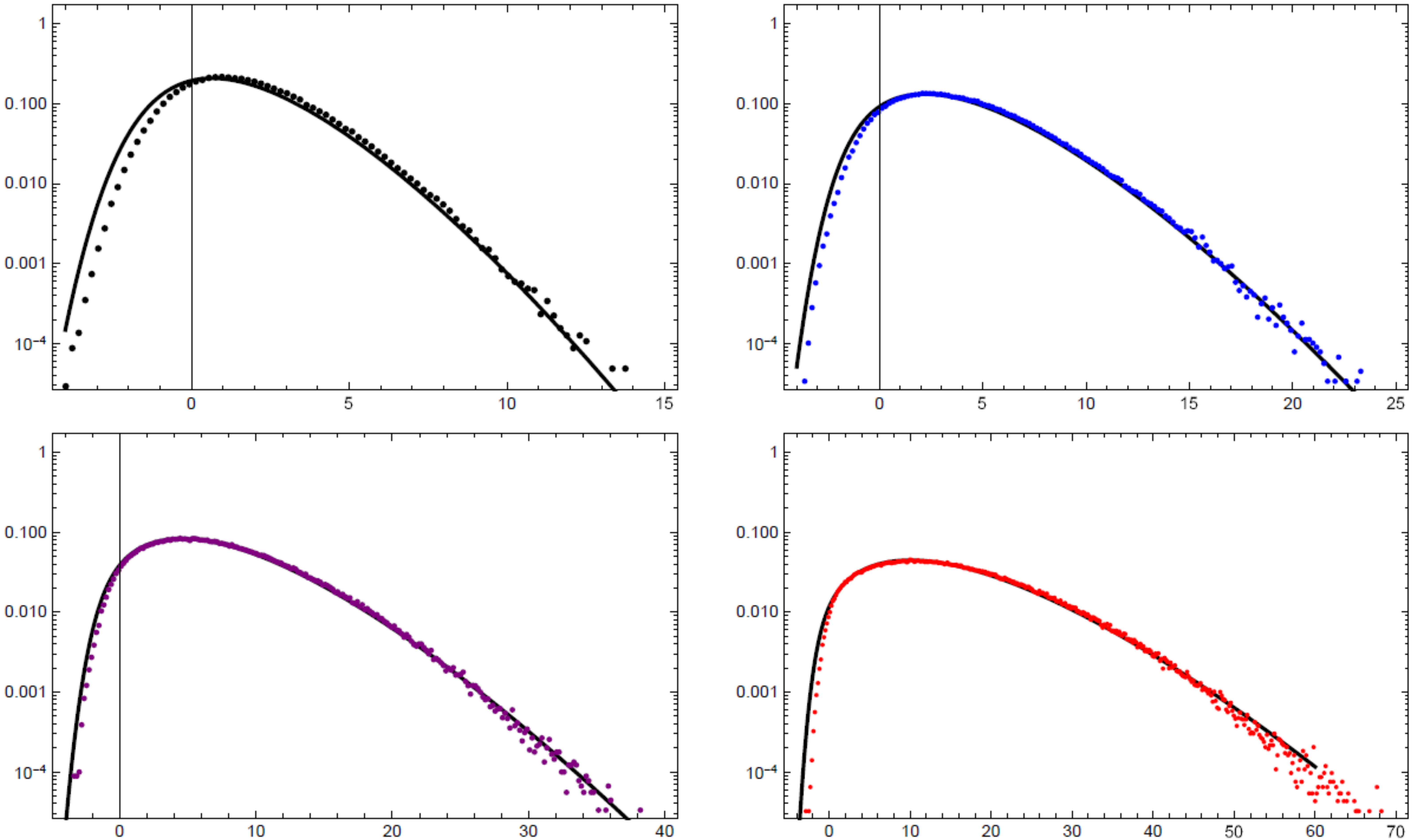
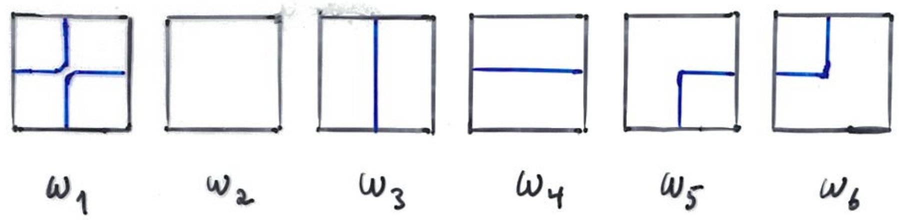
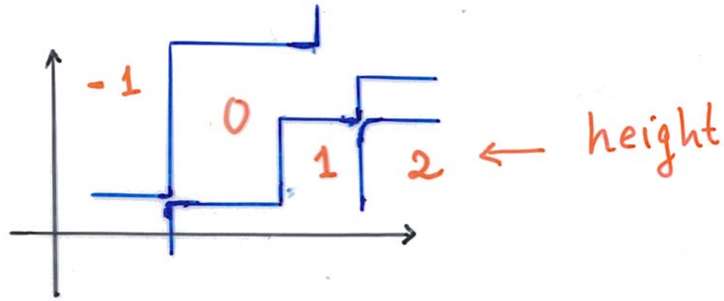


Figure 4: Logarithmic plots of the probability densities of Figure 3.

KPZ point of 6-vertex model



- tiling of the plane
- up-right paths, touching - no crossing



$\Rightarrow$ height function

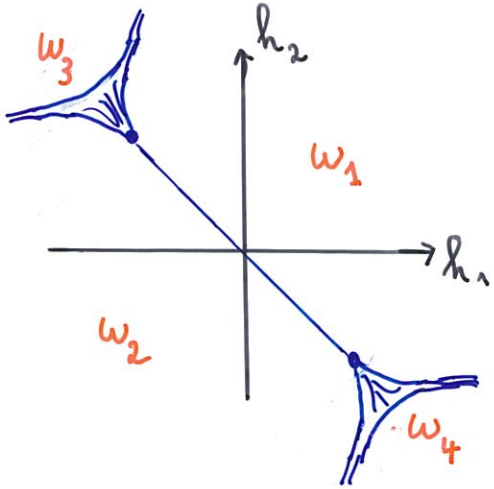
// SM of elastic membrane //

free energy   Bethe ansatz   Lieb, Yang + Yang

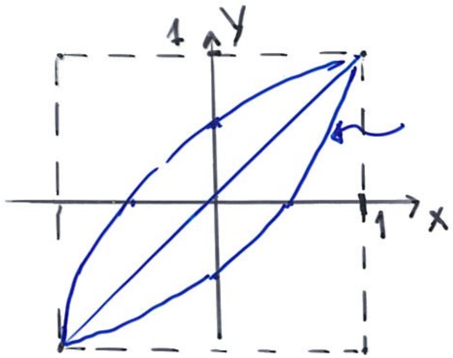
$\Rightarrow$ ferroelectric ($\Delta > 1$)   Bukman, Shore 1995

4 independent parameters

2 temperature-like,   vertical/horizontal electric field $\Leftrightarrow$ polarization



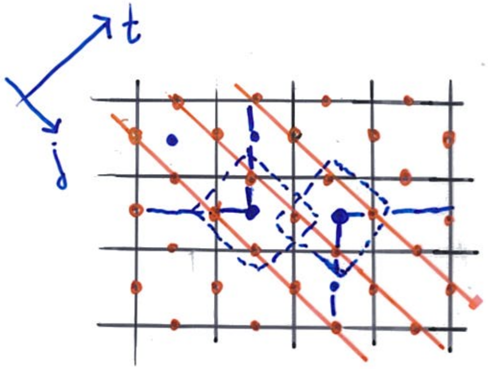
Legendre



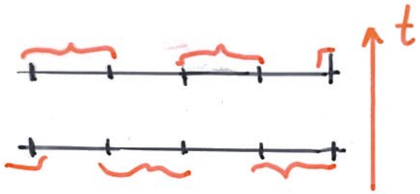
extremal Gibbs measures

- KPZ point $\vec{\omega} = (1, 1, q, p, 1-q, 1-p)$ $\qquad 0 \le p, q \le 1$

diagonal transfer matrix



$$\underbrace{\eta_{ij}}_{\text{edge}} = \begin{cases} 0 & \text{no path} \\ 1 & \text{path} \end{cases}$$



independent update
for each block

1 0	$\rightsquigarrow$	0 1	probability p
0 1	$\rightsquigarrow$	1 0	probability q

implies

0 0	$\rightsquigarrow$	0 0	1
1 1	$\rightsquigarrow$	1 1	1
1 0	$\rightsquigarrow$	1 0	$1-p$
0 1	$\rightsquigarrow$	0 1	$1-q$

Markov chain on $\{0,1\}^{\mathbb{Z}}$

$T = T_{odd} \, T_{even}$

- invariant measure: product period 2 $\mu_{even}(1) = a$, $\mu_{odd}(1) = b$

stationarity $\| (1-q)\, a\, (1-b) = (1-p)(1-a)\, b \|$

η_{ij} stationary, mixing Markov chain $\Leftrightarrow$ extremal Gibbs measure

- initial measure 2-periodic product general a, b

density $\frac{1}{2}$, implies $v = 0$

$\Rightarrow$ non-universal are computed NO adjustable parameters

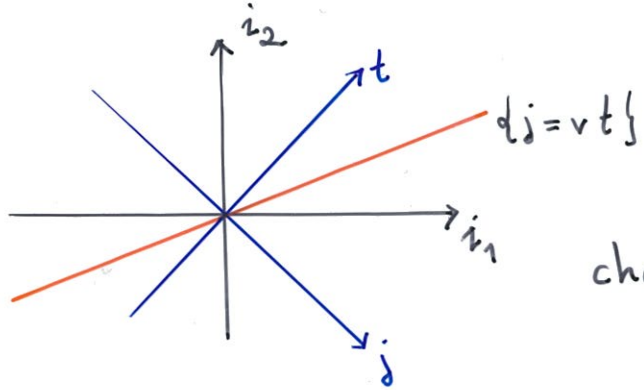
SET $p = 0.6$, $q = 0.2$, $\rho = \frac{1}{2}$ $\sigma = 1$ stationary

$\sigma = 0.65$

$\Rightarrow$ half-infinite 6-vertex $\Leftarrow$

stationary 6-vertex Aggarwal 2016

current-density $j(\rho)$, $v = j'(\rho)$



characteristic

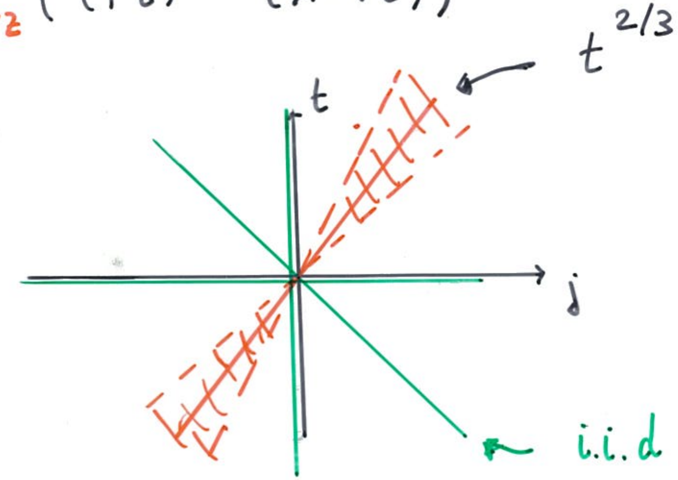
Theorem:

$$h(vt,t) - h(0,0) \cong v_0 t + (\Gamma t)^{1/3} \xi_{BR}$$

and $t^{2/3}$ sidewards

implies two-point

$$S(j,t) = \langle \eta_{jt}\, \eta_{00} \rangle_c \cong A\, (\Gamma t)^{-2/3} f_{KPZ}\big((\Gamma t)^{-2/3} (x - vt) \big)$$



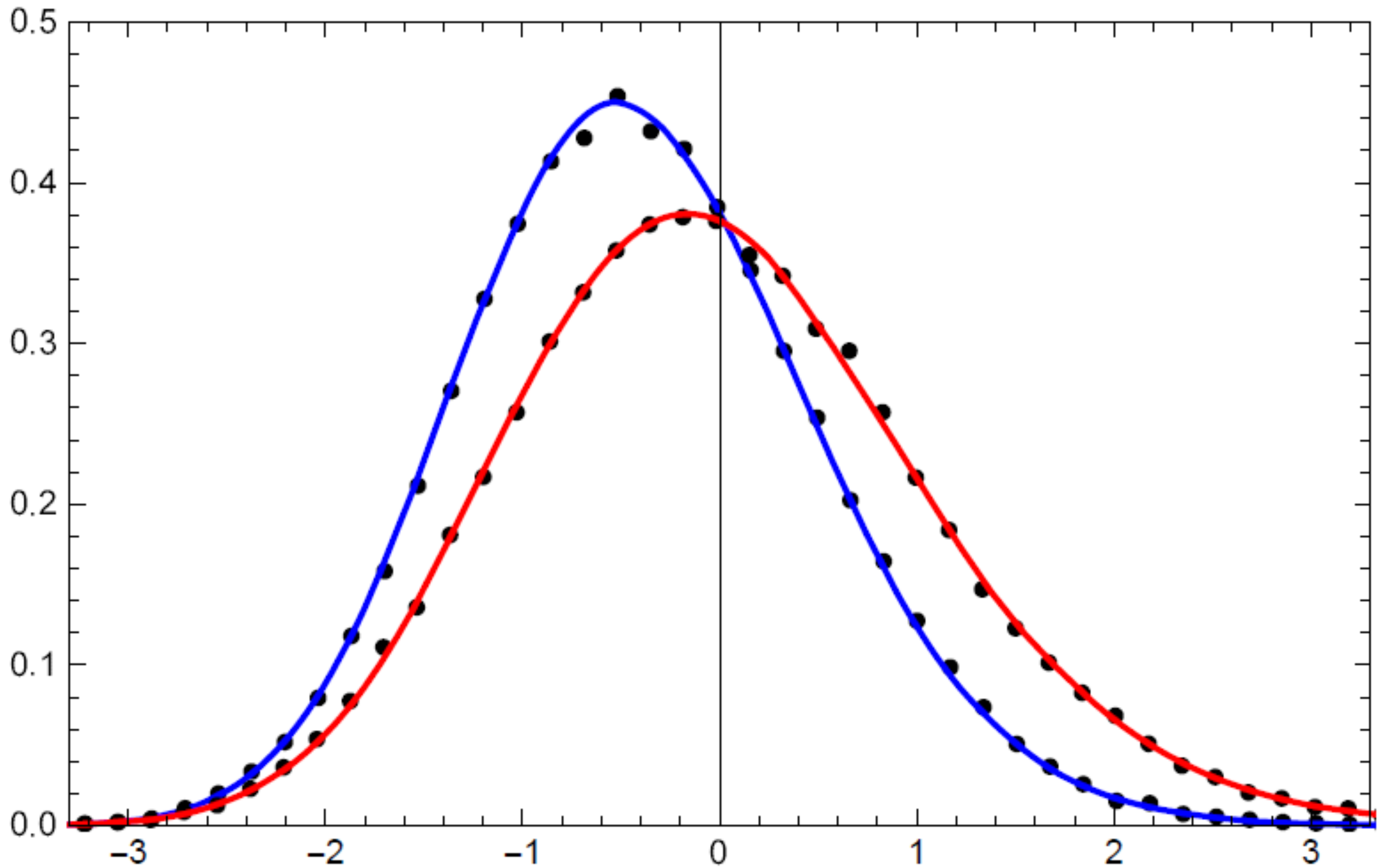


Figure 7: Illustration of the universality: for $\sigma^2 = \alpha/(1-\alpha)$ with $\alpha = 0.3$ (blue) and the stationary benchmark $\alpha = 0.5$ (red), the dots are the probabilities for the current fluctuations of the stochastic six-vertex model. Here we choose generic values, $p = 0.6$, $q = 0.2$, and particle density $\rho = 1/2$. The data are scaled according to KPZ scaling theory.

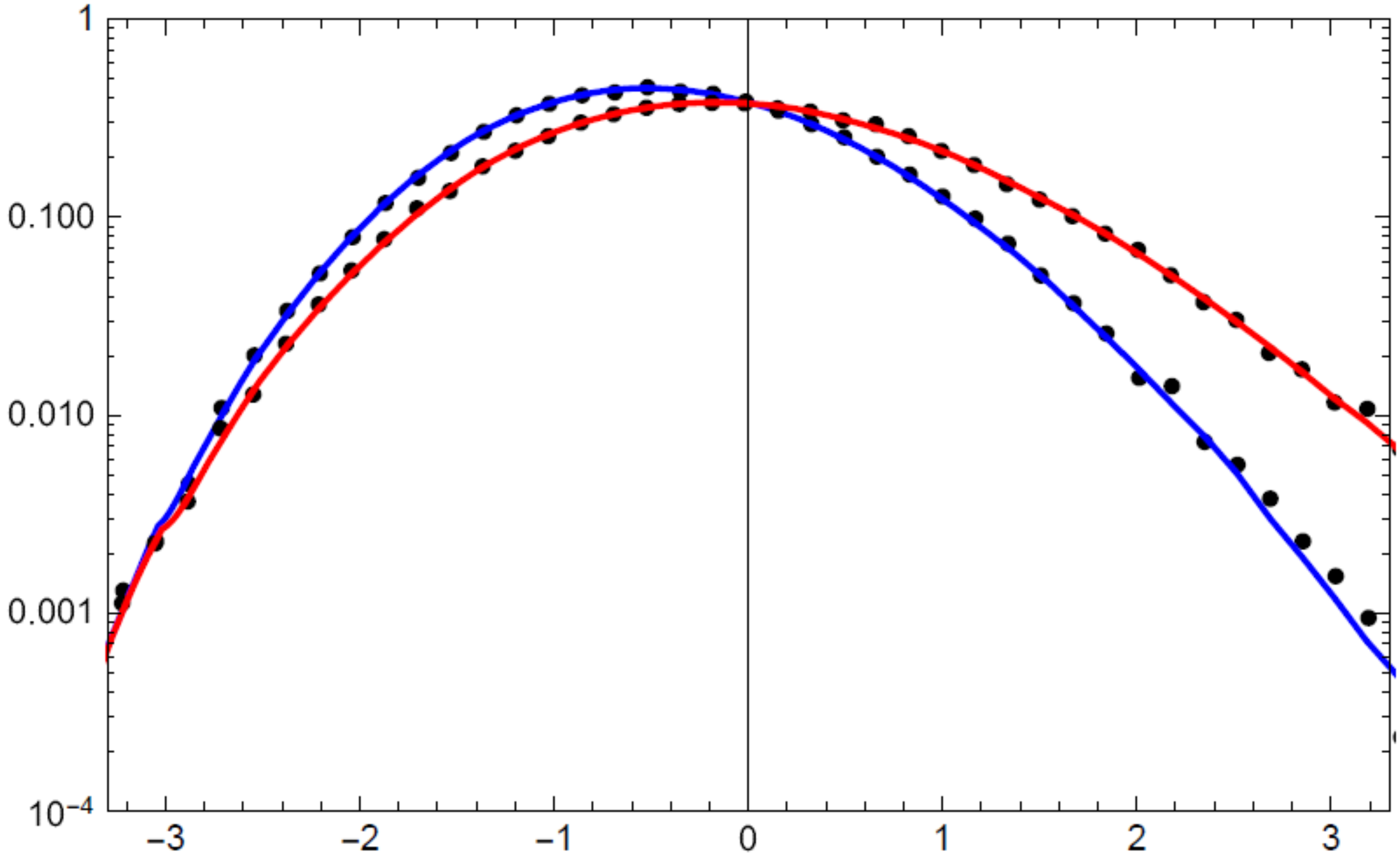


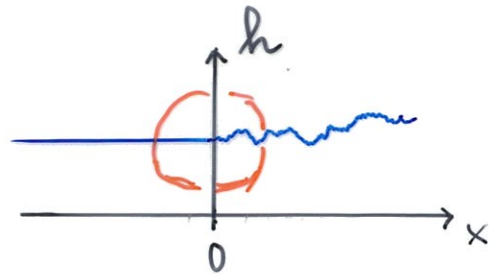
Figure 8: Logarithmic plot of the probability densities of Figure 7.

outlook / discussion

→ dependence on initial conditions

so far: flat, curved, stationary

and crossover



HERE: initial slope is stationary

OPEN PROBLEM: ‖ analytic formuli for $F^{(\sigma)}$, $\sigma \neq 0, 1$ ‖

KPZ fixed point $\lim_{\varepsilon \to 0} \varepsilon^{1/2} h(\varepsilon^{-1} x, \varepsilon^{-3/2} t) = ??$

variational characterization ?