VRJP, random Schrödinger operators and hitting times of Brownian motions

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Explain the relations between

- A family of exchangeable linearly reinforced processes, the VRJP and the ERRW
- A random Schrödinger operator with 1-dependent potential
- Some self-interacting drifted Brownian motion

Note that

- Intimately related to the supersymmetric sigma field of Disertori, Spencer, Zirnbauer.
- Previous works on ERRW : Diaconis Coppersmith, Pemantle, Merkl Rolles, Angel Crawford Kozma

Hitting times of 1D drifted Brownian motion

Let

$$X_t = \theta + B_t - \eta t,$$

where B_t is a standard BM, $\theta > 0$, $\eta \geqslant 0$, and let

$$T = \inf\{t \geqslant 0, \;\; X_t = 0\}.$$

Then, T has an inverse Gaussian law with parameters $(\frac{\theta}{\eta}, \theta^2)$, i.e. has density

$$\mathbb{1}_{t \geqslant 0} \frac{\theta}{\sqrt{2\pi} t^{3/2}} e^{-\frac{1}{2}(\eta^2 t + \theta^2 \frac{1}{t}) + \eta\theta}.$$

Moreover, conditionally on T, $(X_t)_{t \leqslant T}$ has the law of a 3D Bessel bridge from θ to 0 on time interval $[0, T]$.

Vertex Reinforced Random Walk (VRJP)

- $\mathcal{G} = (V, E)$ non-directed graph with conductances $(W_e)_{e \in E}$.
- $(\theta_i)_{i \in V} \in (\mathbb{R}_+^*)^V$ (initial local times).

The VRJP is the continuous time process $(Y_t)_{t \geqslant 0}$ on state space V, that conditionally on $\mathcal{F}_t^Y$, jumps from i to j with rate

$$W_{i,j}(\theta_j + \ell_j^Y(t)),$$

where

$$\ell_i^Y(t) = \int_0^t \mathbb{1}_{Y_s = i} ds$$

is the local time at vertex i.

- Introduced by Davis, Volkov (and proposed by Werner)
- Closely related to the Edge Reinforced Random Walk (ERRW)

Behavior of VRJP and ERRW on $\mathbb{Z}^d$

If $\mathcal{G} = (\mathbb{Z}^d, E_{\mathbb{Z}^d})$ is the d-dimensional network and constant weights ($W_{i,j} = W$ or $a_{i,j} = a$) and $\theta_i = 1$, $\forall i$.

- For any d, the VRJP (resp. ERRW) is positive recurrent at strong reinforcement (i.e. small weights W or a). (S. Tarrès 12, Angel Crawford Kozma 12, (Disertori Spencer))
- For $d \geqslant 3$, the VRJP (resp. ERRW) at weak reinforcement (i.e. large weights W or a) is transient. (S. Tarrès 12, Disertori S. Tarrès 14 (Disertori Spencer Zirnbauer)).

Schrödinger operator : Notations

Let $W = (W_{i,j})_{i,j\in V}$ be the symmetric operator

$$W_{i,j} = \begin{cases} W_e, & \text{if } \{i,j\} = e \in E, \\ 0, & \text{otherwise,} \end{cases}$$

For $(\beta_j)_{j\in V} \in \mathbb{R}^V$, we set

$$H_\beta = 2\beta - W$$

where 2β is the operator of multiplication by $(2\beta_j)_{j\in V}$.

If V finite, we write $H_\beta > 0$ when it is positive definite. In this case H_β^{-1} has positive coefficients (it is an M-Matrix).

Lemma (S., Tarrès, Zeng, 15)

$\mathcal{G}$ finite. Let $(\theta_i)_{i\in V} \in (\mathbb{R}_+^*)^V$.
The following distribution on $\mathbb{R}^V$

$$\nu_V^{W,\theta}(d\beta) = \mathbb{1}_{H_\beta>0} \frac{e^{-\frac{1}{2}\langle\theta, H_\beta\theta\rangle}}{\sqrt{|H_\beta|}} \frac{\prod \theta_i}{\sqrt{2\pi}^{|V|}} d\beta$$

is a probability.
Moreover,

- *$\beta_{|V_1}$ and $\beta_{|V_2}$ are independent if $dist_{\mathcal{G}}(V_1, V_2) \geqslant 2$.*
- *The marginals β_i are such that $\frac{1}{2\beta_i}$ have inverse Gaussian law.*

Lemma (Letac's general version)

$\mathcal{G}$ finite. Let $(\theta_i)_{i\in V} \in (\mathbb{R}_+^*)^V$, and $(\eta_i)_{i\in V} \in (\mathbb{R}_+)^V$. The following distribution on $\mathbb{R}^V$

$$\begin{aligned}\nu_V^{W,\theta,\eta}(d\beta) &:= e^{\langle \eta,\theta\rangle} e^{-\frac{1}{2}\left\langle \eta, H_\beta^{-1}\eta\right\rangle} \nu_V^{W,\theta}(d\beta) \\ &= \mathbb{1}_{H_\beta>0} \frac{e^{-\frac{1}{2}\left\langle \theta, H_\beta\theta\right\rangle - \frac{1}{2}\left\langle \eta, H_\beta^{-1}\eta\right\rangle}}{\sqrt{|H_\beta|}} \frac{e^{\langle \eta,\theta\rangle}\prod_{i\in V}\theta_i}{\sqrt{2\pi}^{|V|}} d\beta\end{aligned}$$

is a probability.

Proposition

If $\beta \sim \nu_V^{W,\theta}$, and $V_1 \subseteq V$, then $\beta_{V_1} \sim \nu_{V_1}^{W,\theta,\eta}$, with

$$\eta = W_{V_1,V_1^c}\theta_{V_1^c}.$$

Theorem (S., Tarrès 12, S. Tarrès Zeng 15)

Let V be finite. Assume that $\beta \sim \nu_V^{W,\theta}$. Let $\delta \in V$ and $V_1 = V \setminus \{\delta\}$. Let $(\psi_j)_{j\in V}$ be the solution of

$$\begin{cases} \psi_\delta = 1, \\ H_\beta(\psi)_{|V_1} = 0, \end{cases}$$

then (after some time change) the VRJP, starting at δ, is a mixture of Markov jump processes with jumping rates

$$\frac{1}{2} W_{i,j} \frac{\psi_j}{\psi_i}$$

- The distribution of $(\psi_i)_{i\in V}$ corresponds to the random field introduced by Disertori, Spencer, Zirnbauer.
- The representation by a random potential $\beta \sim \nu^{W,\theta}$ give a coupling of the representation from different points.

Extension to infinite graphs

$\mathcal{G} = (V, E)$ infinite. Let $(V_n)_{n\in\mathbb{N}}$ be an increasing sequence of subsets of V such that

$$V = \cup_{n\in\mathbb{N}} V_n.$$

By Kolmogorov's extension theorem, we can define the probability distribution $\nu_V^{W,\theta}$ on random potentials $(\beta_i)_{i\in V}$, such that

$$\beta_{V_n} \sim \nu_{V_n}^{W,\theta,\eta^{(n)}},$$

with $\eta^{(n)} := W_{V_n,V_n^c}(\theta_{V_n^c})$,

We now have

$$H_\beta = 2\beta - W \geqslant 0$$

Let $(\psi_j^{(n)})_{j\in V}$ be defined by

$$\begin{cases} \psi^{(n)}_{|V_n^c} = 1, \\ H_\beta(\psi^{(n)})_{|V_n} = 0, \end{cases}$$

Theorem (S., Zeng 15)

Let $\beta \sim \nu_V^{W,\theta}$, and $\mathcal{F}^{(n)} = \sigma\{\beta_i,\ i \in V_n\}$. Then, $\psi_i^{(n)}$ is an $\mathcal{F}^{(n)}$-martingale for all $i \in V$, which converges a.s.

$$\psi_i := \lim_{n\to\infty} \psi_i^{(n)}$$

- *either $\forall i$, $\psi_i = 0$ in which case the VRJP is recurrent.*
- *either $\forall i$, $\psi_i > 0$, in which case the VRJP is transient, and $H_\beta \psi = 0$, and ψ is a generalized eigenstate with eigenvalue 0.*

Let

$$G_\beta(i,j) = H_\beta^{-1}(i,j) + \frac{1}{2\gamma}\psi_i\psi_j,$$

where γ is an independent $\Gamma(\frac{1}{2}, \frac{1}{2})$ random variable. Then, the VRJP starting at i_0 is a mixture of Markov jump processes with rate

$$\frac{1}{2} W_{i,j} \frac{G_\beta(i_0,j)}{G_\beta(i_0,i)}.$$

Consequences

Assume $\mathcal{G}$ is the network $\mathbb{Z}^d$ with $W_e = W$ constant, and $\theta_i = 1$, $\forall i \in V$

- (Disertori, Spencer, Zirnbauer delocalization result) $\Rightarrow$ (for $d \geqslant 3$ and W large enough the martingale is bounded in L^2, hence $\psi_i > 0$ for all $i \in V$).

Theorem (S., Zeng 15)

(i) *In dimension $d \geqslant 3$, at weak reinforcement (W large enough), the VRJP satisfies a functional CLT. (Same for ERRW.)*

(ii) *In dimension $d = 2$, the ERRW is recurrent for any constant weights. (Uses Merkl Rolles 07).*

Hitting times of drifted interacting Brownian motions

For $(T_i)_{i\in V}$ and $t \in \mathbb{R}$ we set

$$K_{t\wedge T} = \mathrm{Id} - (t\wedge T)W \quad \left(= (t\wedge T)H_{\frac{1}{2t\wedge T}}\right)$$

Definition

We consider the S.D.E : let $(B_i(t))_{i\in V}$ be a $|V|$ dim B.M.

$$Y_i(t) = \theta_i + \int_0^t \mathbb{1}_{s<T_i} dB_i(s) - \int_0^t \mathbb{1}_{s<T_i} \left(W\psi(s)\right)_i ds \qquad (1)$$

with

$$\psi(t) = (K_{t\wedge T})^{-1}(Y(t)),$$

and $T_i = \inf\{t \geqslant 0, \quad Y_i(t) - t\eta_i = 0\}$.

The process $(\psi(t))$ is a continuous martingale, more precisely :

$$\psi(t) = \theta + \int_0^t (K_{t\wedge T})^{-1}(\mathbb{1}_{s<T} dB(s)).$$

Theorem (S. Zeng 2017+)

Let $X(t) := Y(t) - (t \wedge T)\eta$.

(i) $\left(\frac{1}{2T_i}\right)$ *has law* $\nu_V^{W,\theta,\eta}$.

(ii) *Conditionally on* (T_i), $(X_k(t))_{0 \leqslant t \leqslant T_k}$ *are independent 3-dimensional Bessel bridges from* θ_k *to* 0.

Proposition (Abelian properties)

(i) *Let* $V_1 \subseteq V$, *then* $X_{V_1}(t)$ *has the same law as the solutions of the S.D.E. on* V_1, *with* W_{V_1,V_1}, θ_{V_1}, *and* $\tilde{\eta} = \eta_{V_1} + W_{V_1,V_1^c}(\theta_{V_1})$.
In particular, marginals $(X_i(t))$ *are drifted BM.*

(ii) *Similar result for the law of* $X_{V_1}(t)$, *conditioned on* $(X_{V_1^c}(t))_{t \geqslant 0}$.