Local Kesten–McKay law for random regular graphs

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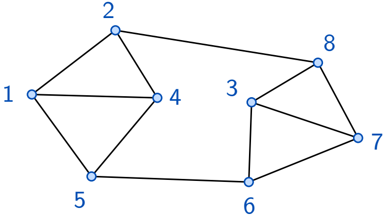
Random regular graph

$\mathsf{G}_{N,d}$ is the set of d-regular graphs on N vertices (without loops and multiedges).

The random regular graph is the uniform probability measure on $\mathsf{G}_{N,d}$.



$$\begin{pmatrix} 0 & 1 & 0 & 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}$$

Adjacency matrix $A_{ij} := \mathbf{1}_{i \sim j}$. Trivial eigenvalue d.

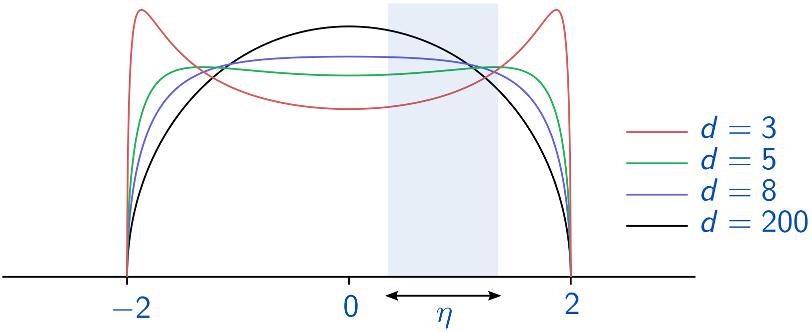
Normalized adjacency matrix $H := A/\sqrt{d-1}$.

Focus in this talk: d fixed (large) and $N \to \infty$. (Earlier results for $d \geqslant (\log N)^4$.)

Background: Kesten–McKay law

For $d \geqslant 3$ fixed, asymptotically almost surely as $N \to \infty$,

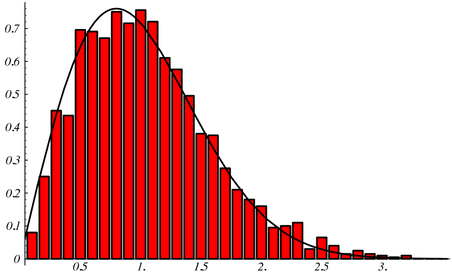
$$\frac{1}{N}\sum_j \delta_{\lambda_j} \longrightarrow \rho_d^{KM}(x) := \left(1 + \frac{1}{d-1} - \frac{x^2}{d}\right)^{-1} \frac{\sqrt{[4-x^2]_+}}{2\pi}\,.$$



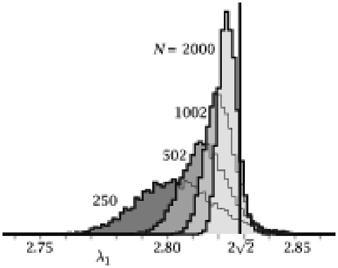
Weak convergence (η fixed).

Background: Numerical evidence for random matrix statistics

Numerical evidence that local eigenvalue statistics of random regular graphs are governed by Gaussian Orthogonal Ensemble (GOE).



Eigenvalue gaps for random 3-regular graphs on 2000 vertices (figure from Jakobson–Miller–Rivin–Rudnick)



Second largest eigenvalue of random 3-regular graphs (figure from Sarnak, What is an Expander)

Numerical results: Jakobson–Miller–Rivin–Rudnick; Newland–Terras; Oren–Smilansky.

Extensive predictions: Smilansky et al.

Results for $d \in [N^{\varepsilon}, N^{2/3-\varepsilon}]$: Bauerschmidt–Huang–Knowles–Yau (eigenvalue gaps).

Background: Eigenvector delocalization

- Random regular graphs have only nonlocalized eigenvectors (with high probability). For example, all eigenvectors v obey

$$\|v\|_\infty = O\left(\frac{1}{(\log N)^c}\right).$$

Stronger results are also known, and such estimates actually hold for regular graphs under the deterministic assumption of local tree-like structure (later).

References: Brooks–Lindenstrauss, Dumitriu–Pal, Geisinger.

- The eigenvectors v of the GOE are uniform on the sphere and therefore (whp)

$$\|v\|_\infty = O\left(\sqrt{\frac{\log N}{N}}\right).$$

Similar estimates are now also known for much more general random matrices.

Example result: Eigenvector delocalization

Background: the eigenvectors v of the GOE satisfy (whp)

$$\|v\|_\infty = O\left(\sqrt{\frac{\log N}{N}}\right).$$

Theorem (Bauerschmidt–Huang–Yau 2016)

Fix $d \geqslant 10^{40}$. Then the eigenvectors v of a random d-regular graph satisfy (whp)

$$\|v\|_\infty = O\left(\frac{(\log N)^{100}}{\sqrt{N}}\right),$$

simultaneously for all eigenvectors v with eigenvalues $|\lambda| < 2\sqrt{d-1} - \varepsilon$.

Preliminaries: Green's function

Green's function (resolvent): $G_{ij}(z) = (H - z)^{-1}_{ij}$ for $z \in \mathbb{C}_+ = \{\operatorname{Im} z > 0\}$.

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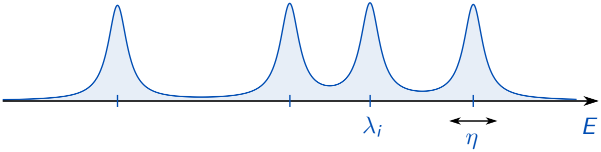
- Spectral density:

$$\frac{\operatorname{Im} \operatorname{Tr}(G(E + i\eta))}{N} =$$



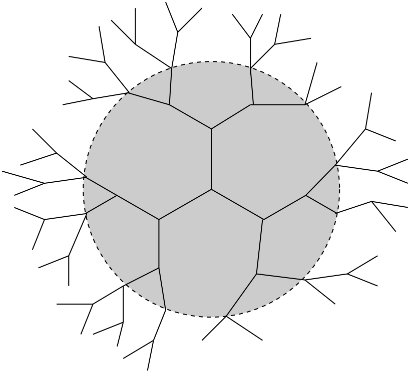
- Eigenvectors: if $Hv = Ev$ then

$$\|v\|_\infty \leqslant \sqrt{\eta \max_i \operatorname{Im} G_{ii}(E + i\eta)} \qquad \text{for any } \eta > 0.$$

- We are interested in $z = E + i\eta$ close to the spectrum: $\eta \approx 1/N$.

Preliminaries: Local geometric structure of random regular graph

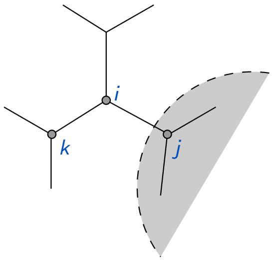
In a random d-regular graph, up to radius $R = c \log_d N$, with high probability:



Most R-balls have no cycles. All R-balls have few cycles.

Preliminaries: Green's function on the infinite regular tree

- Let $G(z) = (A/\sqrt{d-1} - z)^{-1}$ be the Green's function of the d-regular tree;
- and $G^{(i)}(z)$ be that the graph from which vertex i is removed.



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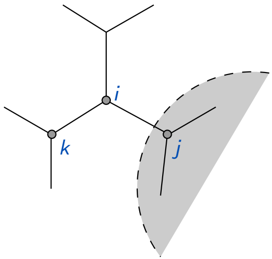
$$G_{ii}(z) = -\left(z + \frac{1}{d-1} \sum_{j \in \partial i} G^{(i)}_{jj}(z) \right)^{-1}$$



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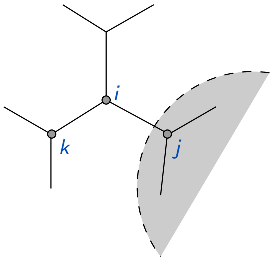
$$G_{ii}(z) = -\left(z + \frac{1}{d-1}\sum_{j\in\partial i} G_{jj}^{(i)}(z)\right)^{-1}$$

$$G_{jj}^{(i)}(z) = -\left(z + \frac{1}{d-1}\sum_{k\in\partial j\setminus i} G_{kk}^{(j)}(z)\right)^{-1}$$



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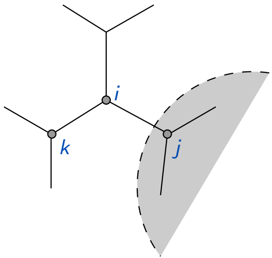


Since $G^{(i)}_{jj}(z)$ is independent of $i \sim j$, the second equation closes:

$$G^{(i)}_{jj}(z) = m_{sc}(z), \quad \text{where } m_{sc}(z) = -\Big(z + m_{sc}(z)\Big)^{-1},$$

and

$$G_{ii}(z) = m_d(z), \quad \text{where } m_d(z) = -\left(1 + \frac{d}{d-1} m_{sc}(z)\right)^{-1}.$$

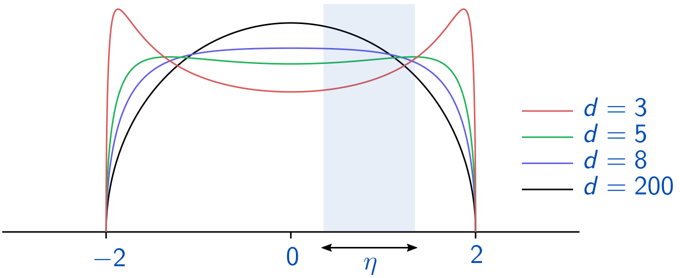
Preliminaries: Kesten–McKay law

Locally tree-like structure → Kesten–McKay law: For $z \in \mathbb{C}_+$ fixed,

$$\frac{1}{N}\operatorname{Tr}(G(z)) \to m_d(z).$$

Equivalent:

$$\frac{1}{N}\sum_j \delta_{\lambda_j} \longrightarrow \rho_d^{KM}(x) := \left(1 + \frac{1}{d-1} - \frac{x^2}{d}\right)^{-1} \frac{\sqrt{[4-x^2]_+}}{2\pi}\,.$$



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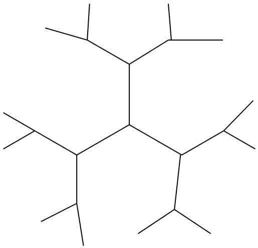


For the infinite tree, the off-diagonal elements can be computed in the same way:

$$G_{ij}(z) = m_d(z) \left(-\frac{m_{sc}(z)}{\sqrt{d-1}} \right)^{\mathrm{dist}(i,j)} .$$

Here $|m_{sc}(z)| \leqslant 1$ and $|m_d(z)| \leqslant 1 + O(1/d)$ uniformly in $z \in \mathbb{C}_+$. In particular, the Green's function decays exponentially and is approximately given by

$$G_{ij}(z) \approx (d-1)^{-\mathrm{dist}(i,j)/2}.$$

Preliminaries: Local semicircle law

For random Wigner matrices (i.i.d. entries):

- Semicircle law (Wigner): For $z \in \mathbb{C}_+$ fixed (whp)

$$\frac{1}{N} \operatorname{Tr}(G(z)) \to m_{sc}(z).$$

- Local semicircle law (Erdős–Schlein–Yau): For $\operatorname{Im} z \gg 1/N$ (whp)

$$\max_i |G_{ii}(z) - m_{sc}(z)| \ll 1. \qquad (+)$$

$(+)$ implies for example that any eigenvector v satisfies

$$\|v\|_\infty \leqslant \sqrt{(\operatorname{Im} z)(m_{sc}(z) + o(1))} \approx \frac{1}{\sqrt{N}}$$

and that spectral density is concentrated on scale $\gg 1$. Such estimates are fundamental in understanding local statistics (Erdős–Schlein–Yau; Tao–Vu).

$(+)$ is not true for random d-regular graphs with fixed d (even with $m_{sc} \rightsquigarrow m_d$).

Local and global structure

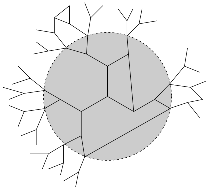
General intuition:

- Random regular graphs are locally given by (almost deterministic) tree-like graphs which are glued together randomly.

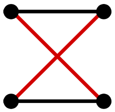
The local structure is given by tree-like neighborhoods.

- Our proof uses that the tree-like structure is valid for radii $\ell \approx \log_d \log N$.
- For $d \gg \log N$, we only need $\ell = 1$: this is the constraint that each vertex has exactly d neighbors.

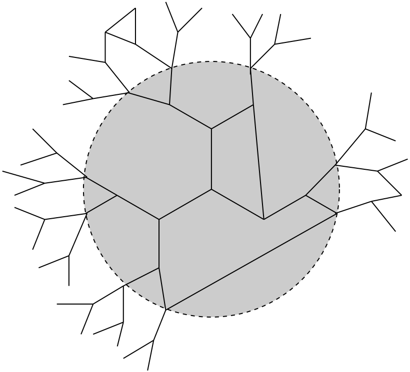
The boundaries of such neighborhoods have $\gg \log N$ edges. This permits the use of concentration estimates.



The global structure is partially captured by invariance (reversibility) under switching dynamics.

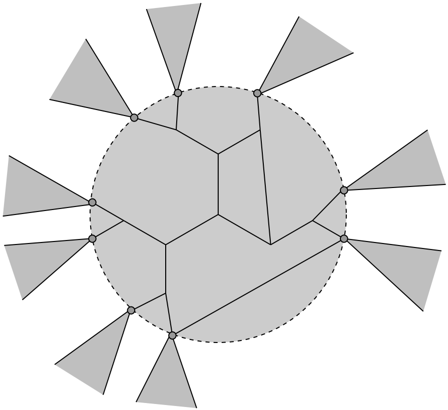


Definition: tree extension



- Recall: most neighborhoods in a random regular graph look locally like a regular tree, but some neighborhoods can have a bounded number of cycles.
- We condition on a large ball $\mathcal{T} = \mathcal{B}_\ell(1, \mathcal{G})$ of radius $\ell = c \log_d \log N$ that has only a bounded number of cycles. Its boundary has size $\Omega(\log N)^c$.

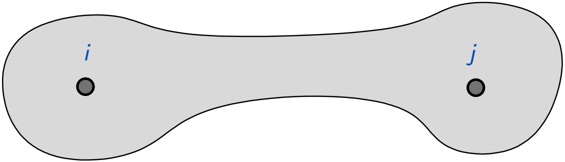
Definition: tree extension



- We replace the graph outside the ball by infinite trees attached to the boundary. We call this graph the tree extension or $\mathsf{TE}(\mathcal{T})$.
- Thus $\mathsf{TE}(\mathcal{T})$ is an infinite graph for which we understand the Green's function very well (expanding the bounded number of cycles about a tree graph).

Main result

Let $\mathcal{E}_r(i,j,\mathcal{G})$ be the union of all paths $i \to j$ of length at most $r = c \log_d \log N$.



Theorem (Bauerschmidt–Huang–Yau 2016)

Fix $d \geqslant 10^{40}$. Then simultaneously for all $z \in \mathbb{C}$ with $\operatorname{Im} z \geqslant N^{-1}(\log N)^{200}$ and $|z \pm 2| \geqslant 1/(\log N)$, and simultaneously for all $i,j \in [\![N]\!]$, with high probability,

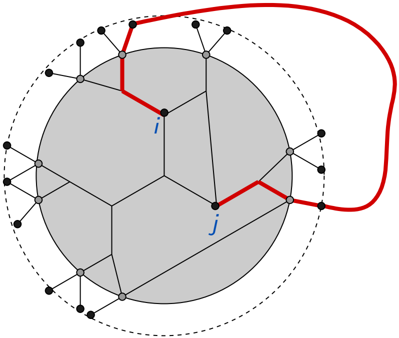
$$G_{ij}(\mathcal{G};z) = \underbrace{G_{ij}(\mathrm{TE}(\mathcal{E}_r(i,j,\mathcal{G});z)}_{P_{ij}(\mathcal{E}_r(i,j,\mathcal{G});z)} + O((\log N)^{-C}).$$

Implies ℓ^∞ eigenvector delocalization and local Kesten–McKay law.

Initial estimates for $\operatorname{Im} z = \Omega(1)$.

- For z far from the spectrum, the random walk picture for $G_{ij}(z)$ is valid.

$$G_{ij}(\mathcal{G}) - P_{ij}(\mathcal{T}) =$$

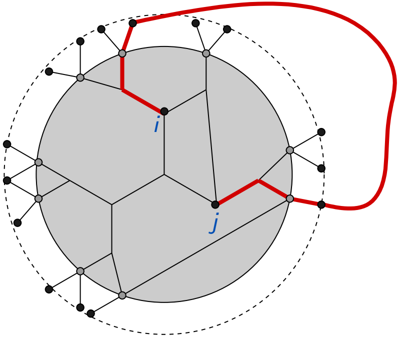


- Decay. For z away from the spectrum, $G_{ij}(z)$ decays exponentially in $\operatorname{dist}(i, j)$ with rate at least $\operatorname{Im} z$: relevant walks are of length less than $1/\operatorname{Im} z$.
- Geometry. Locally tree-like structure implies (deterministically) that most pairs of vertices on the boundary are far from each other.

Multiscale approach for $\operatorname{Im} z \ll 1$.

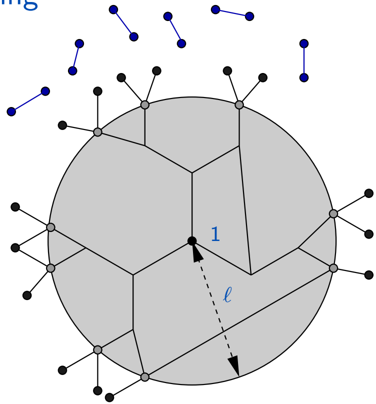
- For $\operatorname{Im} z \ll 1$ the random walk expansion would be highly oscillatory and involve very long paths which we cannot control. Thus we do not use it.

$G_{ij}(\mathcal{G}) - P_{ij}(\mathcal{T}) =$



- Multiscale approach. Use estimates for some $z \in \mathbb{C}_+$ to get same estimates for $z - \varepsilon i$ with slightly lower probability, say $\varepsilon = N^{-3}$. Iterate to $\operatorname{Im} z \approx 1/N$.
- The main difficulty is to get this improvement.

Boundary resampling

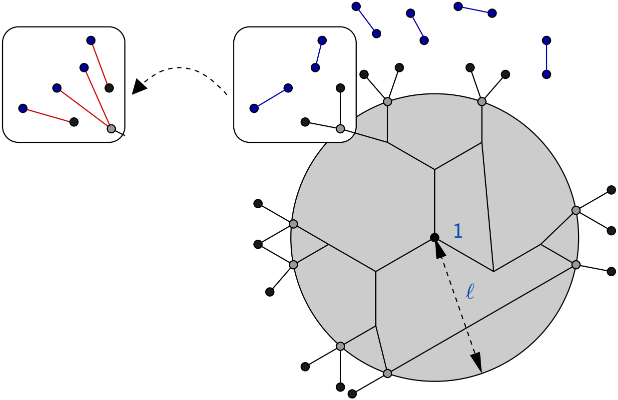


Condition on $\mathcal{B}_\ell(1, \mathcal{G})$.

Pair boundary edges of ball with random edges from graph.

Call this pairing the resampling data $\mathbf{S}$.

Boundary resampling



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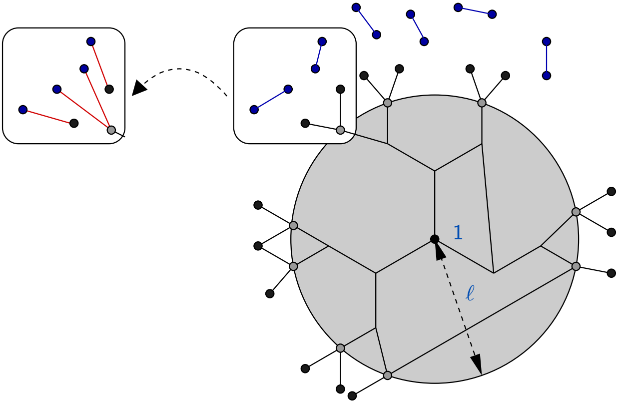
The (simultaneous) switching of all pairs that do not collide with other pairs is measure preserving and actually reversible.

- This defines the switched graph $\tilde{\mathcal{G}} = T_{\mathbf{S}}(\mathcal{G})$.

Boundary resampling



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The (simultaneous) switching of all pairs that do not collide with other pairs is measure preserving and actually reversible.

- This defines the switched graph $\tilde{\mathcal{G}} = T_{\mathbf{S}}(\mathcal{G})$.
- Outer boundary vertices of $\mathcal{B}_\ell(1, \tilde{\mathcal{G}})$ are random under the randomness of $\mathbf{S}$.

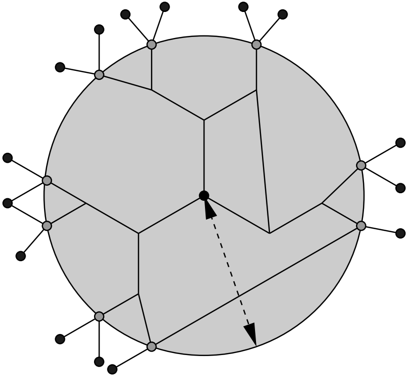
Use of randomness of boundary resampling

We use the randomness of the boundary in the switched graph to obtain two key estimates for the switched graph.

(a) Improved decay of the Green's function between most pairs of distinct boundary vertices.

(b) Concentration of a certain average of the diagonal elements Green's function over the boundary.

These estimates ultimately allow us to obtain a more precise expansion of the Green's function in the interior and advance the iteration.



Use of randomness (a): Improved decay estimate

- For the tree and the tree extensions, we have $P_{ij}(z) \approx (d-1)^{-d(i,j)/2}$.
- Our goal is to prove that the tree extension is accurate up to distance r.
- For distances $> r$, we do not have an estimate better than $(d-1)^{-r/2}$.

Problem:

- There are $\approx (d-1)^{2\ell} = (d-1)^r$ pairs of boundary vertices.

  Even if all paths between these pairs are much longer than r, the previous estimate is not sufficient to bound their contribution.

Upshot:

- We need a better estimate for the Green's function between distinct boundary vertices. We obtain these using that the boundary is essentially random.

Use of randomness (a): Improved decay estimate

Example. Fix a vertex x. Assume $\operatorname{Im} G_{xx}(z) \leqslant 2$ and $\operatorname{Im} z \gg (\log N)^{200}/N$. Then

$$\frac{1}{N}\sum_{b=1}^{N} |G_{xb}(z)|^2 = \frac{\operatorname{Im} G_{xx}(z)}{N \operatorname{Im} z} \ll (\log N)^{-200}.$$

Let $b_1, \ldots, b_\mu$ be independent uniformly random vertices, with say $\mu = (\log N)^5$, independent of the graph. Then (Markov's inequality and union bound)

$$|G_{xb_i}| \leqslant M(\log N)^{-100} \ll d^{-3r/2}, \qquad M = (\log N)^2$$

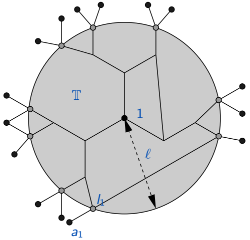
holds for all but at most $\omega' = \log N$ indices i with probability at least

$$\binom{\mu}{\omega'} M^{-2\omega'} \ll N^{-c \log\log N}.$$

Upshot:

- If the boundary $\partial \mathcal{B}_\ell(1, \mathcal{G})$ was independent of the remainder of $\mathcal{G}$, the Green's function between most pairs of vertices would have to be very small.

Use of randomness (b): Concentration estimate



There are μ boundary edges $l_1 a_1, \ldots, l_\mu a_\mu$ in original graph $\mathcal{G}$.

In the switched graph $\tilde{\mathcal{G}}$, the new boundary edges have random endpoints conditioned on $\mathcal{G}$.

Pretend the boundary edges are uniformly random independently of $\mathcal{G}$.

Then

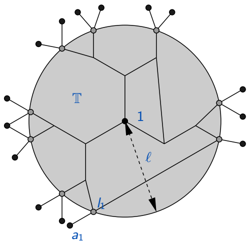
$$\frac{1}{\mu}\sum_{k=1}^{\mu} G_{a_k a_k}^{(l_k)} \quad \text{is the average of } \mu \gg \log N \text{ independent random variables.}$$

Thus it concentrates near its mean

$$Q(\mathcal{G}) = \frac{1}{Nd}\sum_{i\sim j} G_{jj}^{(i)}(\mathcal{G}).$$

For the infinite tree, recall that $G_{jj}^{(i)}(z) = m_{sc}(z)$ solves $m_{sc}^2 + z m_{sc} + 1 = 0$.

Use of randomness (b): Concentration estimate



- But the boundary edges are of course not independent in $\mathcal{G}$.
- In the switched graph $\tilde{\mathcal{G}}$ the new boundary vertices are random and independent conditioned on $\mathcal{G}$.
- Remove conditioned neighborhood $\mathbb{T}$ to remove correlations.

Proposition

Conditioned on $\mathcal{G}$, with high probability in the resampling data,

$$\frac{1}{\mu}\sum_{k=1}^{\mu}\Big(G_{\tilde{a}_k\tilde{a}_k}(\tilde{\mathcal{G}}^{(\mathbb{T})}) - P_{\tilde{a}_k\tilde{a}_k}(\mathcal{E}_r(\tilde{a}_k,\tilde{a}_k,\tilde{\mathcal{G}}^{(\mathbb{T})}))\Big) \approx Q(\tilde{\mathcal{G}}) - m_{sc}$$

Self-consistent estimate

- By resolvent expansion and the decay estimates to bound off-diagonal terms:

$$\tilde{G}_{11} - \tilde{P}_{11} = \frac{1}{d-1} \sum_{k \in [\![1,\mu]\!]} \tilde{P}_{1l_k}^2 (\tilde{G}_{\tilde{a}_k \tilde{a}_k}^{(\mathbb{T})} - \tilde{P}_{\tilde{a}_k \tilde{a}_k}^{(\mathbb{T})}) + O\left(d^{-(r+\ell+2)/2}\right).$$

- If 1 has a tree neighborhood, explicitly $\tilde{P}_{11} = m_d$ and $\tilde{P}_{1l_k} = m_d^2 m_{sc}^{2\ell}/(d-1)^\ell$.
- Use the concentration estimate to replace the sum above:

$$\tilde{G}_{11} - P_{11} = m_d^2 m_{sc}^{2\ell} \frac{d}{d-1} (Q(\tilde{\mathcal{G}}) - m_{sc}) + \text{error}.$$

- Similar (somewhat less explicit) estimates hold if the neighborhood of 1 is not a tree but has (few) cycles.
- Using reversibility this estimate can be pulled back from the switched graph to the original graph. Analogous estimates hold for all other vertices.
- These implies a self-consistent estimate for the averaged quantity $Q(\mathcal{G}) - m_{sc}$.
- This estimate implies the improved estimates for the Green's function.

Induction (simplified)

- Let $\Omega(z) \subset \mathsf{G}_{N,d}$ is a set of graphs which have locally tree-like structure and satisfy the estimate of the main theorem:

$$|G_{ij}(\mathcal{G}; z) - P_{ij}(\mathcal{G}; z)| \leqslant d^{-r/2}.$$

  Initial estimates imply that

$$\mathbb{P}\left(\bigcap_{|z| \geqslant 2d} \Omega(z)\right) = 1 - o(N^{-\omega+\delta}).$$

- Let $\Omega^-(z) \subset \mathsf{G}_{N,d}$ be the set of graphs with a slightly improved bound:

$$|G_{ij}(\mathcal{G}; z) - P_{ij}(\mathcal{G}; z)| \leqslant \frac{1}{2} d^{-r/2}.$$

  Lipschitz continuity of Green's function implies that $\Omega^-(z) \subset \Omega(z - \mathrm{i}/N^3)$.
- Thus it suffices to prove that $\mathbb{P}(\Omega(z) \setminus \Omega^-(z)) \ll N^{-3}$ say.

Induction (simplified)

Fix a radius for the local resampling $r = c \log_d \log N$ and a center vertex, say 1.

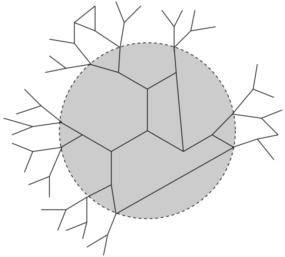
- Define the set $\Omega_1'(z)$ of graphs satisfying improved estimates near 1:

$$G_{1x}(\mathcal{G}; z) = P_{1x}(\mathcal{G}; z) + (\ldots) + O(d^{-(r+1)/2}), \qquad (\ldots)$$

Proposition

For any graph $\mathcal{G} \in \Omega(z)$, with high probability the switched graph is improved near 1:

$$\mathbb{P}\left[T_{\mathbf{S}}(\mathcal{G}) \in \Omega_1'(z) \;\middle|\; \mathcal{G} \right] = 1 - O(N^{-D}).$$



- Reversibility of switchings implies that $\mathbb{P}(\Omega(z) \setminus \Omega_1'(z)) = o(N^{-\omega+\delta})$.
- Union bounds give $\mathbb{P}(\Omega^-(z)) = 1 - o(N^{-\omega+7+\delta})$.

Conclusion

- Random matrix type delocalization estimates for eigenvectors and control of spectral density on all mesoscopic scales for bounded degree regular graphs.
- Simultaneous use of local graph structure and global randomness.
- Method is robust.
- Many interesting questions remain.